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ON A CONVERGENT MESON THEORY. I

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Introduction.

In recent years, important progress in the field of quantum electrodynamics has been obtained by introducing the idea of charge and mass renormalization(1). According to this method, the usual field equations in quantum electrodynamics, which are obtained from the classical equations by a correspondence argument and which contain the well known divergencies, are transformed by a canonical transformation into a set of equations in which only the observable renormalized mass and charge of the particles occur. Since the whole procedure is relativistically invariant one can expect the transformed equations to give a correct description of electromagnetic phenomena, and this expectation has been decisively confirmed by the remarkable accuracy with which this theory allows to calculate the Lamb shift as well as by the predictions of new effects like the anomalous magnetic moment on the basis of the theory.

The application of the renormalization method to the case of nucleons in interaction with mesons seems, however, in some cases to meet with serious difficulties(2). Further, it should be kept in mind that the method itself, in spite of its practical success, is not entirely satisfactory from a theoretical point of view, since the transformation leading to the renormalized equations is not a mathematically well defined unitary transformation, as is obvious from the fact that its purpose is to remove infinities. It would therefore be more attractive, at least in the case of nucleons in interaction with meson fields, to replace the usual field equations by slightly modified equations which, from the beginning, are free of divergencies.

Since the early times of quantum electrodynamics, it has been clear that an essential part of the divergencies inherent in the usual field theories are due to the use of the point particle picture of the elementary particles. Instead of taking the wave functions of the interacting fields at the same space-time point in the interaction Lagrangian, it has, therefore, repeatedly been suggested to introduce a form factor describing a kind of non-localized interaction of the fields[4, 5, 6, 8]. It does not seem possible, however, inside the frame of usual quantum mechanics, to introduce such a form factor in a relativistically invariant way, and for many years all such attempts were regarded as impossible.

In the meantime, the S-matrix theory was developed by HEISENBERG[3]. His starting point was the idea that the framework of ordinary quantum mechanics might be too narrow to comprise a consistent field theory and that the difficulties could be removed only by giving up to some extent the more detailed description of the elementary processes, which is claimed to be possible in the usual quantum mechanics. The directly observable quantities like the cross-sections for the various elementary processes are fully described by the S-matrix and one might take the extreme point of view that a field theory should be considered complete if it only allows of a unique determination fo the S-matrix.

In the present paper, it is shown that the introduction of a form factor in the interaction between particles of spin one half (nucleons) and pseudoscalar mesons leads to a consistent S-matrix theory with correspondence to the usual field theory. Section 1 contains the general formalism including the field equations as well as the expressions for the total energy and momentum of the system. These quantities are in general not constants of the motion but, since they are conserved over infinite time intervals, they may be regarded as constants of collision in the sense of the S-matrix theory. In this section is also given a brief discussion of the general properties of the form factor following from the requirements of relativity, reality, and correspondence. A detailed discussion of the consequences of these requirements is postponed to Section 4.

In the following section, the S-matrix is derived to the second order in the coupling constants by means of the extension of the method of YANG and FELDMAN[10] given by BLOCH[7]. In

Section 3, the expressions for the self-energies of mesons and nucleons are derived from the one-particle part of the S-matrix. The necessary mathematical tools are found in the Appendixes.

The values of the self-energies will, of course, depend on the choice of the form factor and, in Section 4, it is shown that the form factor can be chosen in accordance with the general conditions stated above in such a way that the self-energies are finite and small.

The correspondence requirement implies that the present formalism must be identical with the conventional field theory when the fields are slowly varying. The definition of a slowly varying field involves the introduction of a constant λ of the dimension of a length which also enters into the expression for the form factor in such a way that we get all the results of the usual theory for processes which take place in regions of an extension large compared with λ. This means that, for instance, the second order cross-sections for nucleon-nucleon or nucleon-meson scattering are the same as in the usual meson theory as long as the transferred momentum is smaller than h/λ in the centre of mass system. For high energy processes, however, the form factor causes deviations from the usual theory and, in principle, the results of high energy scattering experiments could be used for an empirical determination of the form factor. At the moment, we have no theory which would allow of a closer determination of the form factor and we do not even know if the constant λ is a universal constant[3 a]. A theory of the present type should perhaps rather be regarded as an approximation to a more general theory applicable to processes in which only particles of the kind considered play an essential role. Hence, the introduction of a form factor may be looked upon as a crude way of taking into account the influence of the external world on the system and it must be expected that the form of the form factor will depend on the particular system considered. Thus, a theory of the form factor itself will require the development of a unified theory of all elementary particles. It is an open question whether this general theory can be developed inside the frame of ordinary quantum mechanics.

In Section 5, some of the most striking differences between the present formalism and ordinary quantum mechanics are discussed, in particular as regards their physical interpretation and the transformation theory. In the present theory, a wider class of transformations—the quasi-canonical transformations—take over the role of the canonical transformations which retain their importance only in the limit of slowly varying fields. It is shown that the theory can be made gauge invariant in the sense that a gauge transformation is equivalent to a quasi-canonical transformation, which means that a gauge transformation has no effect on the physical predictions derived from the theory.

Finally, in Section 6, it is shown that the introduction of the form factor also makes the vacuum polarization finite to the approximation considered. In the present paper, we have discussed the consequences of the theory for scattering processes only. In a subsequent paper, we hope to deal with the properties of composed systems of elementary particles on the basis of this theory. Since the introduction of the form factor effectively means a cut-off, it may be expected that we can avoid the difficulties which, in the usual theory of nuclear forces, arise from the strong singularities of the potentials.

1. General formalism.

In this paragraph, we shall consider the general case of spin one-half particles (nucleons) in interaction with an arbitrary meson field of integer spin. Let $\psi(x)$ be the field variable of the nucleon field, and let the meson field be described by one or several real field variables $u_\alpha(x)$. We assume that the field equations can be derived from a variational principle

$$\delta\left[\int\{L_N(x)+L_M(x)\}\,dx+\int L_{\text{int}}(x',x'',x''')\,dx'\,dx''\,dx'''\right]=0, \qquad (1)$$

where dx is the volume element in Minkowski space, $dx = dx_1\,dx_2\,dx_3\,dx_0$, $x_0 = -i\,x_4$. L_N and L_M refer to the free nucleon and meson fields, respectively, and L_{int} describes the non-localized interaction between the fields. Thus, using units $\hbar = 1$, $c = 1$,

$$\left.\begin{aligned} L_N &= -\left\{\frac{1}{2}(\overline{\psi}\gamma_\mu\partial_\mu\psi-\partial_\mu\overline{\psi}\cdot\gamma_\mu\psi)+M\overline{\psi}\psi\right\} \\ L_M &= -\frac{1}{2}\{\partial_\mu u\cdot\partial_\mu u+m^2u^2\} \end{aligned}\right\} \qquad (2)$$

$$L_{\text{int}} = -\sum_{\zeta'\zeta''\alpha}\overline{\psi}_{\zeta'}(x')\,\Phi^\alpha_{\zeta'\zeta''}(x',x'',x''')\,u_\alpha(x'')\,\psi_{\zeta''}(x'''), \qquad (3)$$

where $\Phi^\alpha_{\zeta'\zeta''}(x', x'', x''')$ in general is a combination of the Dirac matrices depending on three different space-time points. In the following, we shall take Φ as a product of a one-particle matrix operator and a scalar form factor F depending on the coordinates of the three space-time points, i. e.

$$\Phi^\alpha_{\zeta'\zeta''} = \Lambda^\alpha_{\zeta'\zeta''}\cdot F(x',x'',x'''). \qquad (4)$$

For simplicity, we write

$$L_{\text{int}} = -\overline{\psi}(x')\,\Phi(x',x'',x''')\,u(x'')\,\psi(x'''), \tag{5}$$

using vector and tensor notations for the spinor index ζ and the a. Obviously, we have $\Phi u = u\Phi$. The matrix Λ is the usual one-particle operator occurring in the expression for the interaction Langrangian of the corresponding local theory which thus is a special case of the present formalism with

$$F = \delta(x'-x'')\,\delta(x'-x'''). \tag{6}$$

In the case of neutral pseudoscalar mesons, for instance, we have

$$\Lambda F = \left(ig_1\gamma_5 - i\frac{g_2}{m}\gamma_5\gamma_\mu\partial''_\mu\right)F(x',x'',x'''), \tag{7}$$

where g_1 and g_2 are the coupling constants of the pseudo-scalar and the pseudo-vector interactions, respectively. In the case of scalar mesons, we have simply $\Lambda = g\cdot 1$. When we deal with charged and neutral mesons in symmetrical interaction with the nucleons, these expressions should be multiplied by the isotopic spin operator τ^α, $a = 1,2,3$.

As shown by C. Bloch[(7)], Yukawa's theory of non-local fields suggests the following expression for the form factor

$$F = (2\pi)^{-8}\int G(L,l)\exp i\left\{L\left(\frac{x'+x'''}{2}-x''\right)+l(x'-x''')\right\}dL\,dl, \tag{8}$$

where the Fourier transform $G(L,l)$ is a function only of the quantity Π^2 defined by

$$\Pi^2 = l^2 - \frac{(Ll)^2}{L^2}. \tag{9}$$

Here, $Ll = L_\mu l_\mu$ denotes the scalar product of the four-vectors L_μ and l_μ. For time-like L, the Fourier transform G is in this theory given by

$$G(L,l) = G(\Pi) = \frac{\sin\lambda\Pi}{\lambda\Pi}, \tag{10}$$

where λ is a constant of the dimension of a length.

As it will be seen later, the choice of the particular form factor (8), (10) does not lead to a convergent theory. We shall therefore try to develop the theory, as far as possible using a largely arbitrary form factor restricted only by general physical arguments.

In the first place, F must be an invariant by arbitrary displacements of the origin of the system of space-time coordinates. This condition is conveniently expressed in terms of the Fourier transform $F(l^1, l^2, l^3)$ of the function $F(x', x'', x''')$. One sees at once that this condition requires $F(l^1, l^2, l^3)$ to contain a factor $\delta(l^1+l^2+l^3)$. Accordingly, putting $F(l^1, l^2, l^3) = G(l^1, l^3)\,\delta(l^1+l^2+l^3)$, we get

$$\left.\begin{aligned} F(x', x'', x''') &= (2\pi)^{-8}\int G(l^1, l^3) \\ &\cdot \exp i\left\{l^1 x' + l^3 x''' - (l^1+l^3)\,x''\right\} dl^1 dl^3. \end{aligned}\right\} \quad (11)$$

Next, $\int L_{\text{int}}\, dx' dx'' dx'''$ must be Hermitian, which requires

$$F(x', x'', x''') = F^*(x''', x'', x'). \quad (12)$$

In the Fourier representation, this is expressed by

$$G(l^1, l^3) = G^*(-l^3, -l^1). \quad (13)$$

Obviously, as F has to be an invariant with respect to Lorentz transformations, G must have the same property. Sometimes it is convenient to introduce new variables of integration

$$L = l^1 + l^3, \quad l = \frac{l^1 - l^2}{2} \quad (14)$$

into the expression (11), which gives

$$\left.\begin{aligned} F(x', x'', x''') &= (2\pi)^{-8}\int G(l^1, l^3) \\ &\cdot \exp i\left\{L\left(\frac{x'+x'''}{2} - x''\right) + l\,(x' - x''')\right\} dL\,dl. \end{aligned}\right\} \quad (15)$$

In this form, F appears as a function of the variables

$$R = \frac{x'+x'''}{2} - x'', \quad r = x' - x'''. \quad (16)$$

While the dependence of F on the variable r describes a type of internal coupling of the nucleon field to itself, which has no classical analogue, the dependence on R is just what one would expect from analogy with a classical theory of extended interaction between the two types of particles. It will appear, however, that the dependence on r is actually most essential for the convergence of the theory.

Finally, it must be required that the theory of non-localized interaction is equivalent to the usual field theory for sufficiently slowly varying fields. This means that the form factor must have the same effect as the δ-functions of the local theory in any expressions containing slowly varying fields, only.

In Section 4, we shall give a precise definition of what we understand by slowly varying fields as well as a detailed discussion of the restrictions imposed on the form factor from the correspondence requirement mentioned above. The definition of slowly varying fields involves the introduction of a new constant λ into the theory, which conveniently may be taken of the dimension of a length and which one would expect to be of the order of magnitude of, or smaller than, the range of nuclear forces. It will appear that the function F can be chosen to depend on λ in such a way that the limiting cases of $\lambda \to 0$ and of slowly varying fields become identical. Hence, we have

$$\lim_{\lambda \to 0} F(x', x'', x''') = \delta(x' - x'')\,\delta(x' - x''') \tag{17}$$

and, for instance,

$$\int u(x'')\,\psi(x''')\,F(x', x'', x''')\,dx''dx''' = u(x')\,\psi(x') \tag{18}$$

for slowly varying u and ψ.

The equations of motion obtained from the variational principle (1) are

$$\left.\begin{aligned} (\gamma_\mu \partial'_\mu + M)\,\psi(x') &= -\int \Phi(x', x'', x''')\,u(x'')\,\psi(x''')\,dx''dx''' \\ (\Box'' - m^2)\,u(x'') &= \int \overline{\psi}(x')\,\Phi(x', x'', x''')\,\psi(x''')\,dx'dx'''. \end{aligned}\right\} \tag{19}$$

On account of the non-local character of the interaction, the four-current

$$i\overline{\psi}(x)\,\gamma_\mu \psi(x) \tag{20}$$

does not satisfy the continuity equation. In fact, by the usual procedure, one obtains from the first equation (19)

$$\left.\begin{aligned}\partial_\mu (i\overline{\psi}\gamma_\mu\psi) = -i\Big\{\iint \overline{\psi}(x)\,\Phi(x,x'',x''')\,u(x'')\,\psi(x''')\,dx''\,dx''' \\ -\int \overline{\psi}(x')\,\Phi(x',x'',x)\,u(x'')\,\psi(x)\,dx'dx''\Big\}.\end{aligned}\right\} \quad (21)$$

Integrating this equation over the whole four-dimensional space, the right hand side vanishes identically. Hence, we get

$$\int \psi^\dagger(x)\,\psi(x)\,d^{(3)}\vec{x}\,\Big|_{t=-\infty} = \int \psi^\dagger(x)\,\psi(x)\,d^{(3)}\vec{x}\,\Big|_{t=+\infty}. \quad (22)$$

The quantity $\Delta N = \int \psi^\dagger\,\psi\,d^{(3)}\vec{x}$ is equal to the difference between the total number of nucleons and antinucleons, and equation (22) demonstrates that this number is strictly conserved over infinitely large time intervals. This is in general not the case for finite time intervals, where a conservation theorem holds in the limit of slowly varying fields, only. In fact, in this limit we may apply (18) on the right hand side of (21) and the two terms cancel.

The situation is somewhat similar in the case of energy and momentum conservation. The invariance of the Lagrangian with respect to displacements of the origin of the system of space-time coordinates leads again only to the identification of constants of collision. So far treating the field variables as c-numbers we obtain by the usual procedure

$$\left.\begin{aligned}\int \partial_\nu\, t^{(0)}_{\mu\nu}\,\partial x - \iint\Big\{\partial'_\mu\,\overline{\psi}(x')\cdot\Phi(x',x'',x''')\,u(x'')\,\psi(x''') \\ +\overline{\psi}(x')\,\Phi(x',x'',x''')\,\partial''_\mu\,u(x'')\,\psi(x''') \\ +\overline{\psi}(x')\,\Phi(x',x'',x''')\,u(x'')\,\partial'''_\mu\,\psi(x''')\Big\}\,dx'dx''dx''' = 0,\end{aligned}\right\} \quad (23)$$

where $t^{(0)}_{\nu\mu}$ is the usual energy-momentum tensor of the free fields*)

$$\left.\begin{aligned}t^{(0)}_{\mu\nu} = \frac{1}{2}\Big[(\overline{\psi}\gamma_\nu\,\partial_\mu\psi - \partial_\mu\overline{\psi}\cdot\gamma_\nu\psi) - \delta_{\mu\nu}\,(\overline{\psi}\gamma_\lambda\partial_\lambda\psi - \partial_\lambda\overline{\psi}\cdot\gamma_\lambda\psi)\Big] \\ -\delta_{\mu\nu}M\overline{\psi}\psi + \partial_\nu u\cdot\partial_\mu u - \frac{1}{2}\,\delta_{\mu\nu}\,(\partial_\lambda u\cdot\partial_\lambda u + m^2u^2).\end{aligned}\right\} \quad (24)$$

This result can also be verified directly from the equations (19), from which it follows that

*) For quantized fields, the term $\partial_\nu u\partial_\mu u$ should of course be replaced by the Hermitian expression $\frac{1}{2}\,(\partial_\nu u\partial_\mu u + \partial_\mu u\partial_\nu u)$, which involves a corresponding change in the second term on the right hand side of (25).

$$\left.\begin{aligned}\partial_\nu t^{(0)}_{\mu\nu}(x) = &\int \partial_\mu \overline{\psi}(x)\cdot \Phi(x,x'',x''')\, u(x'')\, \psi(x''')\, dx''\, dx''' \\ &+ \int \overline{\psi}(x')\, \Phi(x',x,x''')\, \partial_\mu u(x)\, \psi(x''')\, dx'\, dx''' \\ &+ \int \overline{\psi}(x')\, \Phi(x',x'',x)\, u(x'')\, \partial_\mu \psi(x)\, dx'\, dx''.\end{aligned}\right\} \tag{25}$$

Integrating this equation over the whole domain of four-dimensional space, one obtains again (23). The invariance of the interaction Lagrangian density (3) now allows (23) to be written in the form of an integral of a four-dimensional divergence of a certain tensor $t_{\mu\nu}$. From the invariance of, for instance,

$$\mathcal{L}^{(2)} = -\int \overline{\psi}(x')\, \Phi(x',x,x''')\, u(x)\, \psi(x''')\, dx'\, dx''' \tag{26}$$

and from the fact that the form factor is form-invariant, it follows that

$$\left.\begin{aligned}\partial_\mu \mathcal{L}^{(2)} = -\int \Big\{ &\partial'_\mu \overline{\psi}(x')\cdot \Phi(x',x,x''')\, u(x)\, \psi(x''') \\ &+ \overline{\psi}(x')\;\; \Phi(x',x,x''')\, \partial_\mu u(x)\, \psi(x''') \\ &+ \overline{\psi}(x')\;\; \Phi(x',x,x''')\, u(x)\, \partial'''_\mu \psi(x''') \Big\}\, dx'\, dx'''\end{aligned}\right\} \tag{27}$$

and, hence, (23) can be written in the form

$$\int \partial_\nu t^{(2)}_{\mu\nu}(x)\, dx = 0, \tag{28}$$

where

$$t^{(2)}_{\mu\nu}(x) = t^{(0)}_{\mu\nu}(x) + \delta_{\mu\nu}\mathcal{L}^{(2)}(x). \tag{29}$$

Thus, the following Hermitian quantities are constants of collision

$$G_\mu = -i\int t^{(2)}_{\mu 4}(x)\, d^{(3)}\vec{x} = G^{(0)}_\mu - i\,\delta_{\mu 4}\int \mathcal{L}^{(2)}(x)\, d^{(3)}\vec{x} \tag{30}$$

and may be interpreted as the total momentum and energy of the field. If we had chosen, instead of (26), one of the two other possible interaction Lagrangian densities,

$$\left.\begin{aligned}\mathcal{L}^{(1)} &= -\int \overline{\psi}(x)\, \Phi(x,x'',x''')\, u(x'')\, \psi(x''')\, dx''\, dx''' \\ \mathcal{L}^{(3)} &= -\int \overline{\psi}(x')\, \Phi(x',x'',x)\, u(x'')\, \psi(x)\, dx'\, dx''\end{aligned}\right\} \tag{31}$$

we would, instead of (28), have obtained

$$\int \partial_\nu t_{\mu\nu}^{(1)} dx = 0, \qquad \int \partial_\nu t_{\mu\nu}^{(3)} dx = 0, \tag{32}$$

respectively, where

$$t_{\mu\nu}^{(1)} = t_{\mu\nu}^{(0)} + \delta_{\mu\nu} \mathscr{L}^{(1)}, \qquad t_{\mu\nu}^{(3)} = t_{\mu\nu}^{(0)} + \delta_{\mu\nu} \mathscr{L}^{(3)}. \tag{33}$$

However, the requirement that the energy-momentum tensor must be Hermitian reduces the number of possible choices of this tensor to one of the two

$$\left.\begin{aligned} t_{\mu\nu} &= \frac{1}{2}(t_{\mu\nu}^{(1)} + t_{\mu\nu}^{(3)}) = t_{\mu\nu}^{(0)} + \delta_{\mu\nu} \frac{1}{2}(\mathscr{L}^{(1)} + \mathscr{L}^{(3)}) \\ t_{\mu\nu}^{(2)} &= t_{\mu\nu}^{(0)} + \delta_{\mu\nu} \mathscr{L}^{(2)}. \end{aligned}\right\} \tag{34}$$

It may be remarked that any of these becomes identical with the usual expression of the energy-momentum tensor in the limits of $\lambda \to 0$ or of slowly varying fields.

From the preceding discussion it is clear that the present formalism is entirely different from the Hamiltonian scheme of ordinary quantum mechanics. This is obvious from the fact that the non-local quantities corresponding to the total energy and momentum of the system are in general not constants of motion. However, the fact that these quantities are conserved over the *infinite* time interval $-\infty \leqq t \leqq +\infty$ suggests that G_μ may be regarded as constants of collision in the sense of the S-matrix theory and that a consistent treatment of this formalism can be found inside the frame of HEISENBERG's S-matrix theory. The present theory thus offers an example of a case in which the S-matrix may be calculated without any reference to an underlying Hamiltonian scheme.

2. Derivation of the *S*-matrix.

For the derivation of the S-matrix we shall use the method developed by YANG and FELDMAN and by KÄLLÉN[(10)]. As shown in the Appendix A, the field equations (1.11) are equivalent to the integral equations

$$\left.\begin{aligned} \psi(x) &= \psi(x,\sigma) + \int S_M^\sigma(x,x')\,\Phi(x',x'',x''')\,u(x'')\,\psi(x''')\,dx'\,dx''\,dx''' \\ u(x) &= u(x,\sigma) - \int \overline{\psi}(x')\,\Phi(x',x'',x''')\,\Delta_m^\sigma(x,x'')\,\psi(x''')\,dx'\,dx''\,dx''', \end{aligned}\right\}\quad(1)$$

where $\psi(x,\sigma)$ and $u(x,\sigma)$ are the free fields coinciding with $\psi(x)$ and $u(x)$ on a space-like surface σ. S_M^σ and Δ_m^σ are Green's functions defined by (A. 3) and (A. 32) and corresponding to the mass values M and m, respectively. Taking σ in the infinite past, the functions S_M^σ and Δ_m^σ become identical with the corresponding retarded Green's functions and the equations (1), in this limit, are

$$\left.\begin{aligned} \psi(x) &= \psi^{\mathrm{in}}(x) + \int S_M^{\mathrm{ret}}(x-x')\,\Phi(x',x'',x''')\,u(x'')\,\psi(x''')\,dx'\,dx''\,dx''' \\ u(x) &= u^{\mathrm{in}}(x) - \int \overline{\psi}(x')\,\Phi(x',x'',x''')\,\Delta_m^{\mathrm{ret}}(x-x'')\,\psi(x''')\,dx'\,dx''\,dx'''. \end{aligned}\right\}\quad(2)$$

These equations may be considered as definitions of the in-fields ψ^{in} and u^{in}. As a consequence of (A. 7),

$$\left.\begin{aligned} \psi^{\mathrm{in}}(x) &= \lim_{\sigma\to-\infty} \psi(x,\sigma) \\ u^{\mathrm{in}}(x) &= \lim_{\sigma\to-\infty} u(x,\sigma) \end{aligned}\right\}\quad(3)$$

and the in-fields satisfy the free field equations.

Similarly, we may define the out-fields by

$$\left.\begin{aligned} \psi^{\mathrm{out}}(x) &= \lim_{\sigma\to+\infty} \psi(x,\sigma) \\ u^{\mathrm{out}}(x) &= \lim_{\sigma\to+\infty} u(x,\sigma) \end{aligned}\right\}\quad(4)$$

or, alternatively, by the equations

$$\left.\begin{aligned} \psi(x) &= \psi^{\mathrm{out}}(x) + \int S_M^{\mathrm{adv}}(x-x')\,\Phi(x',x'',x''')\,u(x'')\,\psi(x''')\,dx'\,dx''\,dx''' \\ u(x) &= u^{\mathrm{out}}(x) - \int \overline{\psi}(x')\,\Phi(x',x'',x''')\,\Delta_m^{\mathrm{adv}}(x-x'')\,\psi(x''')\,dx'\,dx''\,dx'''. \end{aligned}\right\}\quad(5)$$

Hence, in a certain sense, the in- and out-fields may be regarded as the free fields which coincide with the actual fields at $t = -\infty$ and $t = +\infty$, respectively, thus representing the ingoing and outgoing fields. By solving the equations (2) we obtain the actual fields in terms of the in-fields. Further, subtracting (5) from (2), we get an expression for the out-fields in terms of the in-fields and the actual fields and, eventually, in terms of the in-fields. Using (A. 20), the equations obtained from (5) and (2) are

$$\left.\begin{aligned} \psi^{\text{out}}(0) &= \psi^{\text{in}}(0) - \int S_M(0-1)\,\Phi(1,2,3)\,u(2)\,\psi(3)\,d(123) \\ u^{\text{out}}(0) &= u^{\text{in}}(0) + \int \overline{\psi}(1)\,\Phi(1,2,3)\,\Delta_m(0-2)\,\psi(3)\,d(123), \end{aligned}\right\} \quad (6)$$

where we use the symbols 0, 1, 2, 3, ... for x, x', x'', x''', ... and $d(123) = dx'\,dx''\,dx'''$.

Following YANG and FELDMAN[10] and BLOCH[7], the quantization of the field variables can now be performed by introduction of commutation relations for the in-field variables. It is then clear from the preceding discussion that also the commutation relations for the actual fields and the out-fields are determined. Since the in-fields satisfy the homogeneous wave equations, we may consistently assume the usual free field commutation relations to hold for these fields, viz.

$$\left.\begin{aligned} \left\{\psi_\zeta^{\text{in}}(x),\ \overline{\psi}_{\zeta'}^{\text{in}}(x')\right\} &= \frac{1}{i} S_{M\zeta\zeta'}(x-x') \\ \left[u^{\text{in}}(x),\ u^{\text{in}}(x')\right] &= i\Delta_m(x-x'). \end{aligned}\right\} \quad (7)$$

It has been shown by BLOCH[7] that then also the out-fields satisfy the commutation relations (7). Consequently, the in- and out-fields must be connected by a unitary transformation

$$\left.\begin{aligned} \psi^{\text{out}} &= S^{-1}\,\psi^{\text{in}}\,S \\ u^{\text{out}} &= S^{-1}\,u^{\text{in}}\,S \\ S^\dagger S &= S S^\dagger = 1. \end{aligned}\right\} \quad (8)$$

On account of the interpretation of the ψ^{out}, u^{out} and ψ^{in}, u^{in} as the variables describing the outgoing and ingoing fields, respectively, the unitary matrix S is the Heisenberg S-matrix of the system[9a].

It is convenient to introduce a Hermitian matrix η by

$$S = e^{i\eta} \tag{9}$$

and the problem is now to determine η from (8) and (9) and the field equations. To solve the field equations we have to take recourse to an iteration method in which the in-fields are chosen as the zero order approximation, and we shall take into account the interaction to the second order in the coupling constants contained in the function Φ. To the first order, we find from (6)

$$\left.\begin{aligned} \psi^{\text{out}}(0) &= \psi^{\text{in}}(0) - \int S_M(0-1)\,\Phi(1,2,3)\,u^{\text{in}}(2)\,\psi^{\text{in}}(3)\,d(123) \\ u^{\text{out}}(0) &= u^{\text{in}}(0) + \int \psi^{\text{in}}(1)\,\Phi(1,2,3)\,\Delta_m(0-2)\,\psi^{\text{in}}(3)\,d(123). \end{aligned}\right\} \tag{10}$$

However, on account of the conservation of energy and momentum, no real first order processes can occur. Consequently, the first order term in η and, therefore, also the first order corrections to the out-fields, must be zero. This can also easily be verified directly by evaluation of the integrals on the right hand side of (10) in momentum space. Therefore, since

$$\left.\begin{aligned} S_M^{\text{ret}} &= \overline{S}_M - \frac{1}{2} S_M \\ \Delta_m^{\text{ret}} &= \overline{\Delta}_m - \frac{1}{2}\Delta_m \end{aligned}\right\} \tag{11}$$

the actual fields calculated to the first order from (2) may be written

$$\left.\begin{aligned} \psi(0) &= \psi^{\text{in}}(0) + \int \overline{S}_M(0-1)\,\Phi(1,2,3)\,u^{\text{in}}(2)\,\psi^{\text{in}}(3)\,d(123) \\ u(0) &= u^{\text{in}}(0) - \int \overline{\psi}^{\text{in}}(1)\,\Phi(1,2,3)\,\overline{\Delta}_m(0-2)\,\psi^{\text{in}}(3)\,d(123). \end{aligned}\right\} \tag{12}$$

Using (12) in (6) we finally get the expressions for the out-fields in terms of the in-fields to the second order in the coupling constants

$$\left.\begin{aligned}
\psi^{\text{out}}(0) &= \psi^{\text{in}}(0) - \int S_M(0-1)\,\Phi(1,2,3)\,u^{\text{in}}(2)\,\bar{S}_M(3-4) \\
&\quad \times \Phi(4,5,6)\,u^{\text{in}}(5)\,\psi^{\text{in}}(6)\,d(1\ldots 6) \\
&\quad + \int S_M(0-1)\,\Phi(1,2,3) \\
&\quad \times \{\bar{\psi}^{\text{in}}(4)\,\Phi(4,5,6)\,\bar{\Delta}_m(2-5)\,\psi^{\text{in}}(6)\}\,\psi^{\text{in}}(3)\,d(1\ldots 6) \\
u^{\text{out}}(0) &= u^{\text{in}}(0) + \int \bar{\psi}^{\text{in}}(1)\,\Phi(1,2,3)\,\Delta_m(0-2) \\
&\quad \times \bar{S}_M(3-4)\,\Phi(4,5,6)\,u^{\text{in}}(5)\,\psi^{\text{in}}(6)\,d(1\ldots 6) \\
&\quad + \int \bar{\psi}^{\text{in}}(4)\,\Phi(4,5,6)\,u^{\text{in}}(5) \\
&\quad \times \bar{S}_M(6-1)\,\Phi(1,2,3)\,\Delta_m(0-2)\,\psi^{\text{in}}(3)\,d(1\ldots 6).
\end{aligned}\right\} \quad (13)$$

Since the first order term in η vanishes, the connection between the in-fields and out-fields expressed by (8) and (9) can, to the second approximation, be written

$$\left.\begin{aligned}
\psi^{\text{out}} &= \psi^{\text{in}} + \frac{1}{i}[\eta, \psi^{\text{in}}] \\
u^{\text{out}} &= u^{\text{in}} + \frac{1}{i}[\eta, u^{\text{in}}].
\end{aligned}\right\} \quad (14)$$

Comparing (13) and (14), and using the free field commutation relations (7) for the in-fields, it is easily verified that the η-matrix in this approximation is given by

$$\left.\begin{aligned}
\eta = &-\int \bar{\psi}^{\text{in}}(1)\,\Phi(1,2,3)\,u^{\text{in}}(2)\,\bar{S}_M(3-4) \\
&\qquad \times \Phi(4,5,6)\,u^{\text{in}}(5)\,\psi^{\text{in}}(6)\,d(1\ldots\ldots 6) \\
&+ \frac{1}{2}\int \bar{\psi}^{\text{in}}(1)\,\Phi(1,2,3) \\
&\quad \times \{\bar{\psi}^{\text{in}}(4)\,\Phi(4,5,6)\,\bar{\Delta}\quad(2-5)\,\psi^{\text{in}}(6)\}\,\psi^{\text{in}}(3)\,d(1\ldots\ldots 6).
\end{aligned}\right\} \quad (15)$$

In this approximation,

$$i\eta = S - 1. \quad (16)$$

According to the S-matrix theory[(3),(9)], the matrix $S-1$ is a product of two factors, the first of which is a δ-function taking care of the conservation of energy and momentum while the square of the second directly gives the cross-sections for the possible real processes.

3. Calculation of the matrix elements of η.

Since the η-matrix given by (2.15) contains in-fields only we shall, in this section, omit the subscript "in" attached to the field variables, and $\overline{\psi}$, ψ and u then denote free field wave functions satisfying the commutation relations (2.7). These functions may in a relativistically invariant way be decomposed into positive and negative frequency parts which then, in the usual way, are interpreted as annihilation and creation operators, respectively. The non-vanishing commutators (anticommutators) between these variables are the following

$$\left.\begin{aligned} \left\{\psi_\zeta^{(+)}(x),\ \overline{\psi}_{\zeta'}^{(-)}(x')\right\} &= -iS_{\zeta\zeta'}^{(+)}(x-x') \\ \left\{\psi_\zeta^{(-)}(x),\ \overline{\psi}_{\zeta'}^{(+)}(x')\right\} &= -iS_{\zeta\zeta'}^{(-)}(x-x') \\ \left[u_\alpha^{(+)}(x),\ u_{\alpha'}^{(-)}(x')\right] &= i\delta_{\alpha\alpha'}\Delta^{(+)}(x-x'), \end{aligned}\right\} \tag{1}$$

where, for simplicity, we use the notations S and Δ instead of S_M and Δ_m. For the definitions of the various Green's functions introduced here see Appendix A. Δ-functions referring to the nucleon mass will be explicitly written Δ_M. The vacuum state vector $|0\rangle$ is now defined by

$$\left.\begin{aligned} \psi^{(+)}|0\rangle &= 0 \\ \overline{\psi}^{(+)}|0\rangle &= 0 \\ u^{(+)}|0\rangle &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} \langle 0|\psi^{(-)} &= 0 \\ \langle 0|\overline{\psi}^{(-)} &= 0 \\ \langle 0|u^{(-)} &= 0. \end{aligned}\right\} \tag{2}$$

In the Appendix B, the matrix elements of the various combinations of wave functions occurring in η have been calculated.

From (B.5) it follows that the vacuum expectation value of the nucleon source density occurring in the interaction between nucleons and pseudoscalar mesons is zero. For instance,

$$\langle 0 \,|\, i\overline{\psi}(1)\,\gamma_5\psi(3)\,|\, 0\rangle = i\,\mathrm{Tr}\,\gamma_5\,S^{(-)}(3-1) = 0, \tag{3}$$

where we have used $\mathrm{Tr}\,\gamma_5 = \mathrm{Tr}\,\gamma_5\gamma_\mu = 0$. This is a particularly simple feature of the pseudoscalar theory. In the scalar meson theory, for instance, the necessary vanishing of the vacuum expectation value of the source density creating the meson field would be obtained only by a suitable symmetrization procedure analogous to HEISENBERG's rule in quantum electrodynamics.

We shall now confine ourselves to the case of pseudoscalar neutral mesons in pseudoscalar interaction with the nucleons and our task will be first to derive the various matrix elements of η.

a) *The self-energy of the meson.* As is well known, the η-matrix contains non-vanishing matrix elements corresponding to transitions between two states in which only one meson and no nucleons are present. Denoting the momenta of the mesons in the initial and final states by p' and p'', respectively, one finds that the matrix element in question is of the form

$$\left. -\pi\,\frac{\delta(p'-p'')}{\omega'}\,\delta m^2, \right\} \tag{4}$$

where δm^2 is an invariant constant and ω is defined by

$$\omega = \sqrt{\vec{p}^{\,2} + m^2}. \tag{5}$$

A term of this form would also arise from an additional term in the interaction Lagrangian density

$$\delta L^{\mathrm{int}} = \delta m^2 u^2. \tag{6}$$

Thus, δm^2 must be interpreted as the contribution to the square of the meson mass due to the interaction with the nucleons. In the local theory this contribution turns out to be infinite. However, as it will be shown below, it is possible to choose the form factor in accordance with the general requirements outlined in

Section 2 in such a way that the correction to the meson mass comes out finite and small compared with the actual meson mass.

To calculate δm^2 we consider the one particle part of the matrix element of η between the two states mentioned above with the corresponding state vectors $|\vec{p}'\rangle$ and $|\vec{p}''\rangle$. In the one particle part

$$\langle \vec{p}'' | \eta_{(1)} | \vec{p}' \rangle = \langle \vec{p}'' | (\eta - \langle 0 | \eta | 0 \rangle) | \vec{p}' \rangle \tag{7}$$

the contribution from the vacuum fluctuations, being of no physical significance, has been subtracted. In the case considered, the form factor can, according to (1.7), with $g_2 = 0$ be written

$$\Phi = ig\gamma_5 F(1, 2, 3) \tag{8}$$

and the η-matrix given by (2.15) becomes

$$\eta = \eta_{\mathrm{I}} + \eta_{\mathrm{II}}, \tag{9}$$

where

$$\left.\begin{aligned} \eta_{\mathrm{I}} &= g^2 \int F(1, 2, 3)\, F(4, 5, 6)\, d(1 \ldots 6) \\ &\quad \times \overline{\psi}(1)\, \gamma_5 \overline{S}(3 - 4)\, \gamma_5\, \psi(6)\, u(2)\, u(5) \\ \eta_{\mathrm{II}} &= -\frac{1}{2} g^2 \int F(1, 2, 3)\, F(4, 5, 6)\, d(1 \ldots 6) \\ &\quad \times \overline{\psi}(1)\, \gamma_5 \left[\overline{\psi}(4)\, \gamma_5 \psi(6)\right] \psi(3)\, \overline{\Delta}(2 - 5). \end{aligned}\right\} \tag{10}$$

Since η_{II} does not contribute to (7), we get

$$\left.\begin{aligned} &\langle \vec{p}'' | \eta_{(1)} | \vec{p}' \rangle = g^2 \int F(1, 2, 3)\, F(4, 5, 6)\, d(1 \ldots 6) \\ &\times \langle 0 | \overline{\psi}(1)\, \gamma_5 \overline{S}(3 - 4)\, \gamma_5 \psi(6) | 0 \rangle \langle \vec{p}'' | [u(2)\, u(5)]_{(1)} | \vec{p}' \rangle. \end{aligned}\right\} \tag{11}$$

The nucleon vacuum expectation value can be evaluated, using (B. 5),

$$\left.\begin{aligned} &\langle 0 | \overline{\psi}(1)\, \gamma_5 \overline{S}(3 - 4)\, \gamma_5 \psi(6) | 0 \rangle \\ &\quad = \sum_{\zeta_1 \zeta_6} (\gamma_5 \overline{S}(3 - 4)\, \gamma_5)_{\zeta_1 \zeta_6} \left(-i S^{(1)}_{\zeta_6 \zeta_1}(6 - 1)\right) \\ &\quad = i\, \mathrm{Tr}\, (\gamma_\mu \partial^{(3)}_\mu + M)(\gamma_\nu \partial^{(6)}_\nu - M)\, \overline{\Delta}_M(3 - 4)\, \Delta^{(-)}_M(6 - 1) \\ &\quad = 4\, i\, (\partial^{(3)}_\mu \partial^{(6)}_\mu - M^2)\, \overline{\Delta}_M(3 - 4)\, \Delta^{(-)}_M(6 - 1). \end{aligned}\right\} \tag{12}$$

The one particle part of the meson matrix element is directly found in (B. 26)

$$\left.\begin{aligned}\langle \vec{p}''\,|\,[u(2)\,u(5)]_{(1)}\,|\,\vec{p}'\rangle &= \frac{1}{2}(2\,\pi)^{-3}\frac{1}{\sqrt{\omega''\omega'}}\\ &\times\left\{e^{i(p'5-p''2)}+e^{i(p'2-p''5)}\right\}.\end{aligned}\right\}\quad(13)$$

Inserting (12) and (13) into (11), and using the Fourier expansions of $\overline{\Delta}_M$, $\Delta_M^{(-)}$ (A. 26), (A. 28), and F (1.15), we find that the first of the two parts of the matrix element (11) arising from the first of the two terms in (13) is

$$\left.\begin{aligned}&2\,g^2(2\,\pi)^{-26}\frac{1}{\sqrt{\omega''\omega'}}\int d(1\ldots 6)\,d(l^1l^3l^4l^6)\,dK'dK\\ &\times G(l^1,l^3)\,G(l^4,l^6)\frac{K'K+M^2}{K'^2+M^2}\,\delta(K^2+M^2)\frac{1-\varepsilon(K)}{2}\\ &\times\exp i\left\{l^1 1+l^3 3-(l^1+l^3)\,2+l^4 4+l^6 6-(l^4+l^6)\,5\right\}\\ &\times\exp i\left\{K'(3-4)+K(6-1)+p'5-p''2\right\}.\end{aligned}\right\}\quad(14)$$

Performing the integration over all the variables except K we obtain

$$\left.\begin{aligned}&2\,g^2(2\,\pi)^{(-2)}\frac{\delta(p'-p'')}{\omega'}\int dK\cdot|\,G(K,-K-p')\,|^2\\ &\times\frac{(p'+K)\,K+M^2}{(p'+K)^2+M^2}\,\delta(K^2+M^2)\frac{1-\varepsilon(K)}{2}.\end{aligned}\right\}\quad(15)$$

The other part of (11), arising from the second term in (13), is obtained from (15) by the transformation

$$\left.\begin{aligned}p' &\to -p''\\ p'' &\to -p'.\end{aligned}\right\}\quad(16)$$

Changing the variable of integration K into $-K$, one finds immediately, by means of the symmetry property (1.13) of G, that this part is identical with (15) except for a change of sign in $\varepsilon(K)$. Hence, we get

$$\langle \vec{p}'' \,|\, \eta_{(1)} \,|\, \vec{p}' \rangle = -\pi \frac{\delta(p'-p'')}{\omega'} \delta m^2, \tag{17}$$

where the correction to the square of the meson mass is

$$\left.\begin{aligned} \delta m^2 = & -\frac{2g^2}{\pi}(2\pi)^{-2}\int dK \cdot |\,G(K, -K-p')\,|^2 \\ & \times \frac{(p'+K)K+M^2}{(p'+K)^2+M^2}\,\delta(K^2+M^2). \end{aligned}\right\} \tag{18}$$

In the local limit $G = 1$ and we obtain the well known result that δm^2 is quadratically divergent. We also see that a $G(l^1, l^3)$, depending on $l^1 + l^3$ only, cannot bring about convergence, since in this case the form factor occurring in (18) is independent of the variable of integration. Finally, it is easily seen that the choice (1.10) of the form factor following from Bloch's version of Yukawa's theory only reduces the degree of divergence to a logarithmic one.

b) *The self-energy of the nucleon.* In the same way, we now consider the matrix elements of the one particle part of η corresponding to a transition from a state $|\,\sigma' P'\rangle$ with one nucleon present with wave vector P' and spin σ' to a state $|\,\sigma'' P''\rangle$ and we obtain a result of the form

$$\langle \sigma'' P'' \,|\, \eta_{(1)} \,|\, \sigma' P' \rangle = -2\pi\delta(P''-P')\cdot\{j_\mu \delta A_\mu + I\delta M\}. \tag{19}$$

Here, j_μ and I are defined in terms of the spinor plane wave amplitudes (p. 47) by

$$\left.\begin{aligned} j_\mu &= j_\mu(\sigma'', \sigma'; \vec{P}') = i\bar{v}(\sigma''\vec{P}')\gamma_\mu v(\sigma'\vec{P}') \\ I &= I\;(\sigma'', \sigma'; \vec{P}') = \bar{v}(\sigma''\vec{P}')\,v(\sigma'\vec{P}'). \end{aligned}\right\} \tag{20}$$

Further, δM and δA_μ are a scalar and a four-vector, respectively, given by

$$\left.\begin{aligned} \delta A_\mu &= \delta A_{\mathrm{I}\mu} + \delta A_{\mathrm{II}\mu} \\ \delta M &= \delta M_{\mathrm{I}} + \delta M_{\mathrm{II}}, \end{aligned}\right\} \tag{21}$$

where

$$\left.\begin{aligned}
\delta A_{\mathrm{I}\mu} &= \frac{1}{(2\pi)^3} g^2 \int dk \,|\, G(P', k - P') \,|^2 \\
&\quad \times \frac{(P' - k)_\mu}{(P' - k)^2 + M^2} \delta(k^2 + m^2) \frac{1 + \varepsilon(k)}{2} \\
\delta M_{\mathrm{I}} &= \frac{1}{(2\pi)^3} g^2 \int dk \,|\, G(P', k - P') \,|^2 \\
&\quad \times \frac{M}{(P' - k)^2 + M^2} \delta(k^2 + m^2) \frac{1 + \varepsilon(k)}{2} \\
\delta A_{\mathrm{II}\mu} &= \frac{1}{(2\pi)^3} g^2 \int dK \,|\, G(K, - P') \,|^2 \\
&\quad \times \frac{K_\mu}{(K - P')^2 + m^2} \delta(K^2 + M^2) \frac{1 - \varepsilon(K)}{2} \\
\delta M_{\mathrm{II}} &= \frac{1}{(2\pi)^3} g^2 \int dK \,|\, G(K, - P') \,|^2 \\
&\quad \times \frac{M}{(K - P')^2 + m^2} \delta(K^2 + M^2) \frac{1 - \varepsilon(K)}{2} .
\end{aligned}\right\} \tag{22}$$

A term of the type (19) would appear in the S-matrix from an additional term in the interaction part of the Lagrangian of the form

$$\delta L_{\mathrm{int}} = i \overline{\psi} \gamma_\mu \psi \delta A_\mu + \overline{\psi} \psi \delta M. \tag{23}$$

Such a term corresponds to an additional term in the energy of the free particle field

$$\delta H = \int \psi^* \left\{ - \vec{\alpha}\, \delta \vec{A} + \delta A_0 - \beta \delta M \right\} \psi \, d^{(3)} \vec{x} . \tag{24}$$

Hence, δM should be considered as the contribution to the nucleon mass due to the interaction with the meson field, while $\delta \vec{A}$ and δA_0 represent a constant self-potential.

4. General properties of the form factor. Convergence of the theory to the second order.

In this section, we shall investigate the general properties of the form factor following from the correspondence requirement briefly mentioned in Section 1, and we shall show that it is possible to choose the form factor in accordance with the result of this investigation in such a way that no divergencies occur in the theory to the second order approximation in the coupling constants. It will be our first task to give a precise formulation of what we understand by a slowly varying field. It is clear that a field variable which could be considered slowly varying at one time or, more generally, in the neighbourhood of a space-like surface σ will not retain this property throughout the whole space-time. Accordingly, the definition of the slowly varying field must be given with reference to a certain surface σ. The field variables will now be called slowly varying on σ if, in a suitably chosen Lorentz system, the free field functions $\psi(x,\sigma)$ and $u(x,\sigma)$ which coincide with $\psi(x)$ and $u(x)$ on σ may be regarded as built up of plane waves involving only momenta small compared with $\frac{1}{\lambda}$. In this way, the notion of slowly varying fields is given a relativistically invariant meaning, but it may be remarked that the expression slowly varying then is somewhat misleading, since it is obvious that slowly varying fields are not composed of waves corresponding to small momenta, only, in every Lorentz system.

The correspondence with the local theory now requires that the evolution of the thus defined slowly varying fields in the neighbourhood of the surface σ is the same as in the usual theory. The value of $\psi(x)$ for x on an infinitesimally displaced surface σ' is given by (2.1), and since

$$S^{\sigma}(x,x') \tag{1}$$

is zero if x' is outside the domain in four-space between the neighbouring surfaces σ and σ', the integral on the right hand side of (2.1) is small of the first order in the distance between σ and σ'. Neglecting terms of the second order in this distance, the usual iteration procedure for solving (2.1) gives for x in the neighbourhood of σ

$$\left.\begin{aligned} \psi(x) &= \psi(x,\sigma)+\int S^{\sigma}_{\mathrm{M}}(x,x')\,\Phi(x',x'',x''')\,u(x'',\sigma)\,\psi(x''',\sigma)\,dx'dx''dx''' \\ u(x) &= u(x,\sigma)-\int \overline{\psi}(x',\sigma)\,\Phi(x',x'',x''')\,\Delta^{\sigma}_{\mathrm{m}}(x,x'')\,\psi(x''',\sigma)\,dx'dx''dx'''. \end{aligned}\right\} \text{(2}$$

Comparing the first of these equations with the corresponding local equation we see that the form factor must satisfy the condition

$$\left.\int F(x', x'', x''')\, u(x'', \sigma)\, \psi(x'', \sigma)\, dx'\, dx'' = u(x', \sigma)\, \psi(x', \sigma)\ \right\} \quad (3)$$

for arbitrary, slowly varying ψ and u. Introducing the Fourier expansions of the function F (1.15) in (3), we obtain

$$\left.\begin{aligned} &\int G(P+p, -P)\, u(p, \sigma)\, \psi(P, \sigma)\, e^{i(P+p)x'} dp dP = \\ &\qquad = \int u(p, \sigma)\, \psi(P, \sigma)\, e^{i(P+p)x'} dp dP, \end{aligned}\right\} \quad (4)$$

where $u(p, \sigma)$, $\psi(P, \sigma)$ are the Fourier transforms of $u(x, \sigma)$, $\psi(x, \sigma)$, respectively.

Hence, G must satisfy the condition

$$G(P+p, -P) = 1 \quad (5)$$

whenever P and p are four-momenta entering in the Fourier expansions of the slowly varying ψ and u. From the Hermitian conjugate equation of (4) we get similarly, using (1.13),

$$G^*(l^1, l^3) = G(-l^3, -l^1)$$

and

$$u^*(p) = u(-p),$$

the further condition

$$G(P, p-P) = 1. \quad (6)$$

Finally, the second equation (2) leads to the condition

$$G(P', -P'') = 1 \quad (7)$$

which must hold for any two four-momenta P^{II} and P^{I} occurring in the Fourier expansions of the slowly varying nucleon wave function.

If the form factor G is assumed to be real we have the symmetry relation

$$G(l^1, l^3) = G(-l^3, -l^1) \quad (8)$$

and, since G must be an invariant, it can be a function of the three invariants

$$\left(\frac{l^1 - l^3}{2}\right)^2, \quad (l^1 + l^3)^2, \quad \left[(l^1 + l^3)\left(\frac{l^1 - l^3}{2}\right)\right]^2 \tag{9}$$

only. We shall show, however, that it is possible to obtain convergence with a G depending only on one variable. The calculation of the correction of the meson mass (3.18) shows that this variable cannot be $(l^1 + l^3)^2$. Similarly, the last variable in (9) is excluded since it is constant and in fact zero for the momenta entering in the last term of the expression for the self-energy of the nucleons (3.22). Accordingly, the only possibility left is to choose G as a function of $[(l^1 - l^3)/2]^2$ or a combination of the quantities (9) containing $[(l^1 - l^3)/2]^2$. It was found convenient to choose the combination

$$\Pi^2 = \left(\frac{l^1 - l^3}{2}\right)^2 - \frac{\left[(l^1 + l^3)\left(\frac{l^1 - l^3}{2}\right)\right]^2}{(l^1 + l^3)^2} \tag{10}$$

which is identical with Π^2 entering in (1.10).

For the l^1 and l^3 values in condition (7), we have

$$\Pi^2(l^1, l^3) = \left(\frac{P'' + P'}{2}\right)^2. \tag{11}$$

If P_0'' and P_0' have the same sign, i.e. if P'' and P' are wave vectors corresponding to the same type of particles, $P'' + P'$ is time-like. In the rest system of the two particles, where $\vec{P}'' = -\vec{P}' = \Delta\vec{P}$, Π^2 is

$$\Pi^2 = -\left[\left(\Delta\vec{P}\right)^2 + M^2\right]. \tag{12}$$

On the other hand, if P'' and P' are wave vectors of an antinucleon and a nucleon, $-P'' + P'$ is time-like and in the rest system of the two particles, where now $\vec{P}'' = \vec{P}' = \Delta\vec{P}$, Π^2 is

$$\Pi^2 = \left(\Delta\vec{P}\right)^2. \tag{13}$$

The condition (7) now requires that $G = 1$ for values of Π^2 corresponding to $(\Delta\vec{P})^2$ small compared with $1/\lambda^2$. This suggests the following choice of a simple form factor depending on one variable, only,

$$G(\Pi^2) = 1 \quad \text{for} \quad \left\{ \begin{array}{l} -M^2 - \dfrac{1}{\lambda^2} \leq \Pi^2 \leq -M^2 \\[2ex] \quad 0 \leq \Pi^2 \leq \dfrac{1}{\lambda^2} \end{array} \right. \tag{14}$$

and zero outside these intervals. For the l^1 and l^3 occurring in the conditions (5) and (6), we have

$$\Pi^2 = -M^2 + \frac{(Pp)^2}{m^2}. \tag{15}$$

In the rest system of the nucleon, (15) becomes

$$\Pi^2 = \frac{M^2}{m^2}(\vec{p})^2. \tag{16}$$

Hence, the choice (14) of the form factor is also in accordance with the conditions (5) and (6). However, on account of the factor M/m in (16), we see that with the choice (14) the range of momenta for which we have correspondence to the usual theory is more restricted for the mesons than for the nucleons.

Using the explicit expression (14) for the form factor the self-energies of the meson and the nucleon derived in Section 3 may now be evaluated. The meson self-energy (3.18) contains a G-factor

$$|G(K, -K-p')|^2. \tag{17}$$

In the frame of reference where the meson is at rest, we have

$$\Pi^2 = \vec{K}^2. \tag{18}$$

Accordingly, the form factor restricts the domain of integration to

$$0 \leq \vec{K}^2 \leq \left(\frac{1}{\lambda}\right)^2.$$

In the meson rest system, we obtain from (3.18), performing the intergrations over K_0 and over all directions of $\vec{K}$,

$$\delta m^2 = -\frac{4}{\pi}\left(\frac{g^2}{4\pi}\right)\int_0^{\frac{1}{\lambda}} \frac{\sqrt{k^2+M^2}\cdot k^2\,dk}{k^2+M^2-\frac{m^2}{4}}. \tag{19}$$

Whence, to the first order in $\frac{1}{M\lambda}$

$$\frac{\delta m}{m} = \frac{\delta m^2}{2m^2} = -\frac{2}{3\pi}\left(\frac{g^2}{4\pi}\right)\frac{m}{M}\cdot\alpha^{-3}. \tag{20}$$

$\alpha = m\lambda$ is the ratio between λ and the meson Compton wavelength $\frac{1}{m}$ and may be expected to be of the order of magnitude of unity.

For the nucleon self-energy (3.22) we obtain in the rest system of the particle, in the same approximation as before,

$$\left.\begin{aligned} \frac{\delta M_{\mathrm{I}}}{M} &= -\frac{1}{6\pi}\left(\frac{g^2}{4\pi}\right)\left(\frac{m}{M}\right)^4\alpha^{-3} \\ \delta\vec{A}_{\mathrm{I}} &= 0 \\ \frac{\delta(A_{\mathrm{I}})_0}{M} &= \frac{\delta M_{\mathrm{I}}}{M} \end{aligned}\right\} \tag{21}$$

and

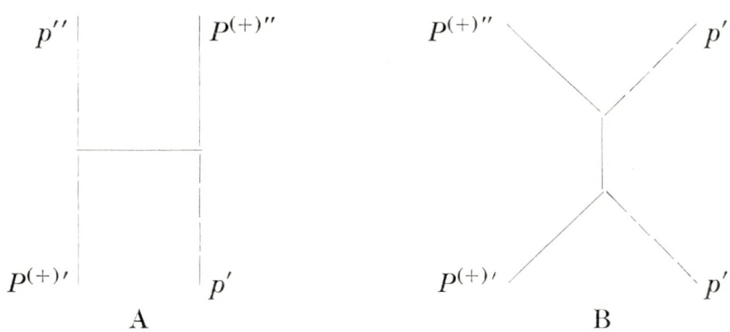
$$\left.\begin{aligned} \frac{\delta M_{\mathrm{II}}}{M} &= -\frac{2}{3\pi}\left(\frac{g^2}{4\pi}\right)\left(\frac{m}{M}\right)^3\alpha^{-3} \\ \delta\vec{A}_{\mathrm{II}} &= 0 \\ \frac{\delta(A_{\mathrm{II}})_0}{M} &= \frac{\delta M_{\mathrm{II}}}{M}. \end{aligned}\right\} \tag{22}$$

Introducing for m the mass of the π-meson, and putting $g^2/4\pi \sim \frac{1}{10}$, we obtain

$$\left.\begin{aligned} \frac{\delta m}{m} &\sim 10^{-2}\,\alpha^{-3} \\ \frac{\delta m}{m} &\sim 10^{-4}\,\alpha^{-3} \end{aligned}\right\} \tag{23}$$

which, for α of the order of magnitude of unity, means that the mass corrections are small fractions of the actual masses.

It is instructive by direct calculation to verify that the form factor (14), which was chosen in accordance with the correspondence requirement formulated in the beginning of this section, actually does not affect the cross-sections for nucleon-nucleon scattering and for the scattering of mesons by nucleons for sufficiently weak collisions. We shall not here give any detailed derivation of the corresponding matrix elements of the η-matrix. The calculation is quite straightforward, and the results will be quoted without proof. In the local limit, the matrix element of η for a transition from an initial state with one meson of momentum p' and one nucleon of momentum $P^{(+)'}$ present to a final state where the particles have momenta p'' and $P^{(+)''}$, respectively, consists of two contributions corresponding to the two graphs



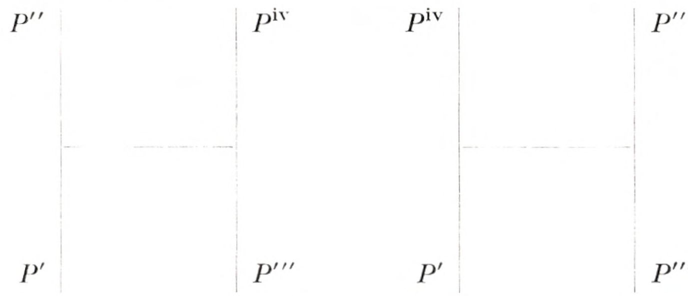
Let the contribution from the first graph be $\langle P^{(+)''} p'' | A | P^{(+)'} p' \rangle$ and that from the second $\langle P^{(+)''} p'' | B | P^{(+)'} p' \rangle$. Then, the corresponding matrix element in the theory of non-localized interaction can be written in the form

$$\left.\begin{aligned}&\langle P^{(+)''} p'' | A | P^{(+)'} p' \rangle G(P^{(+)''}, -P^{(+)''} - p') G(P^{(+)'} + p'', -P^{(+)'})\\ +&\langle P^{(+)''} p'' | B | P^{(+)'} p' \rangle G(P^{(+)''}, -P^{(+)''} - p'') G(P^{(+)'} + p', -P^{(+)'}).\end{aligned}\right\} \quad (26)$$

Also the matrix element determining the nucleon-nucleon scattering cross-section can in the local limit be written as a sum of two terms A and B. If P' and P''' denote the momenta of the incident nucleons, and P'' and P^{iv} those of the scattered nucleons, the corresponding matrix elements in the theory of non-localized interaction are

$$\left.\begin{aligned} &\langle P'' P^{\mathrm{iv}} \,|\, A \,|\, P' P''' \rangle \, G(P'', -P') \, G(P^{\mathrm{iv}}, -P''') \\ +&\langle P'' P^{\mathrm{iv}} \,|\, B \,|\, P'' P''' \rangle \, G(P^{\mathrm{iv}}, -P') \, G(P'', -P''') . \end{aligned}\right\} \quad (27)$$

The two terms are the contributions from the following two graphs, respectively.



By comparing the G-factors in (26) and (27) with those in (5), (6), and (7) it becomes clear from the discussion on p. 26 and 27 that the scattering matrix elements (26) and (27) are identical with those of the corresponding local theory for all processes in which the momenta involved are small compared with $\frac{1}{\lambda}$ in the frame of reference where the center of gravity of the system is at rest.

5. Physical interpretation of the theory. Transformation theory.

In the general formalism developed in Section 1, the variables $\overline{\psi}(x)$, $\psi(x)$ and $u(x)$ play a role similar to that of the field variables in the usual theories, in as far as the connection between these variables in different space-time points is given by certain integro differential equations. However, the physical interpretation of the field variables is in general much more complicated than in the usual theory. In fact, a direct interpretation is given only for the in- and out-fields which are the quantities having a simple physical meaning. In the general case, the ψ and u variables may rather be regarded as a kind of auxiliary quantities giving the connection between the directly observable in- and out-fields and thus allowing of a determination of the S-matrix.

The usual interpretation of the field variables is possible only in the limit of slowly varying fields where the conventional theory is valid.

The present formalism offers an example of a theory which allows the S-matrix to be calculated for any system of interacting nucleons and mesons. The only arbitrariness still present in the theory is that involved in the choice of the invariant function $G(l^1, l^3)$. This function could in principle be determined by comparison of the results of high energy scattering experiments with the cross-sections following from the theory.

In order to obtain a convergent theory, it seems necessary to give up some of the general concepts of quantum mechanics and, to avoid paradoxes, it is important to realize the fundamental difference between a theory of the kind considered here and the usual quantum mechanical description. This difference was strikingly illustrated already in the first section, where it was pointed out that the quantities which in the local limit correspond to energy, momentum, and charge of the system cannot be considered constants of the motion. This should, however, not be considered a defect of the theory, since it is sufficient to require that these quantities in general are constants of collision.

On account of the non-Hamiltonian form of the present formalism it is clear that also the notion of canonical transformations loses its importance in this theory. There are other, more general transformations, however, which play a similar role as the canonical transformations do in ordinary quantum mechanics. In the local theory, a canonical transformation of the field variables $\varphi(x)$ can always be written in the form

$$\mathring{\varphi}(x) = T^{\dagger} \varphi(x) T, \tag{1}$$

where T is a unitary operator which may be regarded as an arbitrary functional of the field variables $\varphi(\vec{x}, t)$ on a space-like surface $t =$ constant. This transformation has the property that the commutation relations for the transformed variables $\mathring{\varphi}$ are the same as those for the old variables on the surface $t =$ constant. Further, the field equations in terms of the new variables have again the form of canonical equations of motion with the same Hamiltonian H, although of course H is a different function of the transformed variables than it is of the old variables.

In an S-matrix formalism where the S-matrix is defined as the unitary matrix connecting two sets of free field variables φ^{in} and φ^{out}, by the equation

$$\varphi^{\text{out}}(x) = S^{\dagger} \varphi^{\text{in}}(x) S, \tag{2}$$

one is led to consider canonical transformations of the in- and out-variables given by

$$\left.\begin{aligned} \mathring{\varphi}^{\text{in}}(x) &= T^{\text{in}\dagger} \varphi^{\text{in}}(x)\, T^{\text{in}} \\ \mathring{\varphi}^{\text{out}}(x) &= T^{\text{out}\dagger} \varphi^{\text{out}}(x)\, T^{\text{out}}, \end{aligned}\right\} \tag{3}$$

where T^{in} and T^{out} are certain functionals of $\varphi^{\text{in}}(x)$ and $\varphi^{\text{out}}(x)$, respectively, on the arbitrary surface $t =$ constant. From (3) we get

$$\mathring{\varphi}^{\text{out}} = \mathring{S}^{\dagger} \mathring{\varphi}^{\text{in}} \mathring{S}, \tag{4}$$

where

$$\mathring{S} = T^{\text{in}\dagger} S T^{\text{out}} \tag{5}$$

is a unitary matrix. If the transformation (3) is such that

$$T^{\text{out}} = S^{\dagger} T^{\text{in}} S, \tag{6}$$

which means that T^{out} is the same functional of out-variables as T^{in} is of the in-variables, we have

$$\left.\begin{aligned} \mathring{S} &= T^{\dagger}[\varphi^{\text{in}}] \cdot S \cdot T[\varphi^{\text{out}}] S^{\dagger} S \\ &= T^{\dagger}[\varphi^{\text{in}}] \cdot T[\varphi^{\text{in}}] \cdot S = S \end{aligned}\right\} \tag{7}$$

and the S-matrix is invariant. A transformation of this kind may be called a "collision transformation" and, in a pure S-matrix theory, such transformations play a similar role as the canonical transformations in the usual theory.

In a formalism like the present, which pretends to link up the pure S-matrix description with the usual quantum mechanical description, a certain class of transformations of the variables φ are of special importance. To any collision transformation corresponds a very wide class of transformations

$$\mathring{\varphi} = \mathring{\varphi}[\varphi(x)] \tag{8}$$

which have the property that $\overset{\circ}{\varphi}(x)$ asymptotically for $t \to \pm\infty$ coincides with $\overset{\circ}{\varphi}^{\text{in}}$ and $\overset{\circ}{\varphi}^{\text{out}}$, respectively. However, on account of the correspondence requirement, we are only interested in those transformations which in the limit of slowly varying fields reduce to canonical transformations. Transformations of this kind will be called quasi-canonical.

We shall now consider a special type of quasi-canonical transformations, viz. the gauge transformation

$$\left.\begin{aligned} \overset{\circ}{\varphi}(x) &= e^{i\chi(x)}\psi(x) \\ \overset{\circ}{u}(x) &= u(x) \end{aligned}\right\} \quad (9)$$

which transforms the field equations (1.19) into

$$\left.\begin{aligned} \{\gamma_\mu(\partial_\mu - i\partial_\mu\chi) + M\}\overset{\circ}{\psi} &= -\int \overset{\circ}{\Phi}(x, x'', x''')\, u(x'')\, \overset{\circ}{\psi}(x''')\, dx''dx''' \\ (\Box - m^2)\, u &= \int \overset{\circ}{\bar{\psi}}(x')\, \overset{\circ}{\Phi}(x', x, x''')\, \overset{\circ}{\psi}(x''')\, dx'dx''', \end{aligned}\right\} \quad (10)$$

where we have put

$$\overset{\circ}{F}(x', x'', x''') = e^{i\chi(x')} F(x', x'', x''')\, e^{-i\chi(x''')}, \quad \overset{\circ}{\Phi} = \Lambda \overset{\circ}{F}. \qquad (11)$$

Since the transformed in- and out-fields are equal to the original in- and out-fields times $e^{i\chi}$, it is clear that the S-matrix connecting the in- and out-fields remains unchanged by this transformation. As is well known, the phase transformation of the free field variables is a canonical transformation of the type (3) with

$$\left.\begin{aligned} T^{\text{in}} &= T[\psi^{\text{in}}] = \exp\left\{i\int \psi^{\text{in}\,\dagger}(\vec{x}, t)\, \psi^{\text{in}}(\vec{x}, t)\, \chi(\vec{x}, t)\, d^{(3)}\vec{x}\right\} \\ T^{\text{out}} &= T[\psi^{\text{out}}]. \end{aligned}\right\} \quad (12)$$

In the case of slowly varying fields, both $\overset{\circ}{\Phi}$ and Φ are effectively equal to δ-functions, and we have complete gauge invariance in the usual sense. On the other hand, if the fields cannot be considered slowly varying, the form factor F must transform along with a phase transformation of the ψ's in accordance with (11).

If the nucleons are protons subject to an external electromagnetic field, a gauge invariant theory can again only be obtained if the form factor is considered as dependent on the four-potentials of the external field. As remarked by C. Bloch, a formally gauge invariant theory can be obtained in the case of an external electromagnetic field if the form factor is taken as

$$\Phi_A(x', x'', x''') = \exp\left(-i\int_{x'}^{x'''} A_\mu\, dx_\mu\right)\cdot \Phi(x', x'', x'''), \tag{13}$$

where Φ is the form factor for $A_\mu = 0$, and the path of integration is taken as the straight line connecting the points x' and x''' in Minkowski space. The field equations can then be taken as

$$\left.\begin{aligned} \{\gamma_\mu(\partial_\mu - ieA_\mu) + M\}\,\psi &= -\int \Phi_A(x, x'', x''')\, u(x'')\, \psi(x''')\, dx'' dx''' \\ (\Box - m^2)\, u &= \int \overline{\psi}(x')\, \Phi_A(x', x, x''')\, \psi(x''')\, dx' dx'''. \end{aligned}\right\} \tag{14}$$

It is easily seen that the so defined form factor by the gauge transformation

$$\overset{\circ}{A}_\mu = A_\mu + \partial_\mu \Lambda \tag{15}$$

of the potentials transforms as

$$\overset{\circ}{F}_A(x', x'', x''') = e^{ie\Lambda(x')}\, F_A(x', x'', x''')\, e^{-ie\Lambda(x''')}, \tag{16}$$

which means that the transformation (16) is equivalent to a quasi-canonical transformation of the type (9).

Instead of this completely gauge invariant scheme with the complicated form factor (14) an alternative procedure would be to fix the gauge of the potentials by choosing these as the retarded potentials from external current and charge distributions. Since the retarded potentials in the limit of vanishing current and charge distribution tend to zero, it would be consistent to choose the same form factor as in the case of no external fields. For a different choice of gauge, the form factor should then be transformed in accordance with (11).

6. The polarization of the vacuum by an external meson field.

As is well known, the coupling of the meson field to the nucleon field in its vacuum state gives rise to a polarization effect which, in the language of perturbation theory, can be attributed to the virtual creation and annihilation of nucleon pairs. In this section, we shall confine ourselves to the approximation where the meson field can be treated as a classical field. Although the physical interpretation of an external meson field is not at all obvious, an investigation of this kind throws some light on the types of polarization effects which are caused by quantized meson fields.

To illustrate this effect, we shall calculate the vacuum expectation value of the source density

$$I(x) = ig\int \overline{\psi}(1)\,\gamma_5 F(1, x, 3)\,\psi(3)\,d(13) \tag{1}$$

to the second order in the coupling constant. To simplify the problem, we only treat the case of a meson field which is weak, in the sense that no real scattering and pair creation processes take place to the first order in the coupling constant g. Consequently, the first order correction to the out-fields obtained from the field equations (2.6) vanishes. Transforming the expression (2.10) for this correction to momentum space, it can be seen that the Fourier components $u(p)$ of a weak meson field vanish whenever pair creation is compatible with the conservation laws of energy and momentum, i. e. whenever

$$p = P - \overline{P}, \tag{2}$$

where P and $\overline{P}$ are nucleon wave vectors, $P^2 = \overline{P}^2 = -M^2$. Hence, the only non-vanishing Fourier components of u are those corresponding to wave vectors which could be considered as four-momenta of a particle with rest mass smaller than $2M$. In the same way as in Section 2, the vanishing of the first order correction to the out-fields allows one to simplify the expression of the first order correction $\psi^{(1)}$ to the ψ's to

$$\psi^{(1)}(0) = ig\int \overline{S}(0-1)\,\gamma_5 F(1, 2, 3)\,u(2)\,\psi^{\mathrm{in}}(3)\,d(123) \tag{3}$$

given by (2.12). We can now calculate the vacuum expectation value of the source density (1). To the second order in the coupling constant g, we get

$$\left.\begin{aligned}\langle I\rangle_0 &= ig\int F(1\,x\,3)\,\langle\,\overline{\psi}^{\text{in}}(1)\,\gamma_5\,\psi^{\text{in}}(3)\,\rangle_0\,d(13)\\ &+ ig\int F(1\,x\,3)\,\langle\,\overline{\psi}^{(1)}(1)\,\gamma_5\,\psi^{\text{in}}(3)\,\rangle_0\,d(13)\\ &+ ig\int F(1\,x\,3)\,\langle\,\overline{\psi}^{\text{in}}(1)\,\gamma_5\,\psi^{(1)}(3)\,\rangle_0\,d(13).\end{aligned}\right\}\tag{4}$$

The first of the terms on the right hand side vanishes. The two other ones can be evaluated using standard methods given in the Appendix B and we obtain after a short calculation

$$\left.\begin{aligned}\langle I\rangle_0 &= -4\,g^2\,(2\,\pi)^{-3}\int dp\,dL\,|\,G\,(L+p,-L)\,|^2\\ &\times\frac{pL}{2\,pL+p^2}\,\delta\,(L^2+M^2)\,u\,(p)\,e^{ipx}.\end{aligned}\right\}\tag{5}$$

Here, G is the Fourier transform of the form factor and $u(p)$ is defined by

$$u\,(x) = \int u\,(p)\,e^{ipx}\,dp.\tag{6}$$

(5) can conveniently be written in the form

$$\langle I\rangle_0 = \int\Phi\,(p)\,u\,(p)\,e^{ipx}dp\tag{7}$$

or, alternatively,

$$\langle I\rangle_0 = \Phi\,(-i\partial_\mu)\,u\,(x).\tag{8}$$

From (5) we obtain the expression for Φ,

$$\Phi = -4\,g^2\,(2\,\pi)^{-3}\int dL\,\frac{|\,G\,(L+p,-L)\,|^2\cdot(pL)\cdot\delta\,(L^2+M^2)}{2\,pL+p^2}.\tag{9}$$

According to our assumption about the external field, p is a time-like vector and we can introduce the variable of integration Λ defined as the magnitude of $\vec{L}$ in the frame of reference where the "meson is at rest", i.e. where $p = \left(\vec{0},\pm i\sqrt{-p^2}\right)$. Using (4.18), and performing three of the integrations, we obtain

$$\left.\Phi = -4\,g^2\,(2\,\pi)^{-2}\int_0^\infty d\Lambda\,|\,G\,(\Lambda^2)\,|^2\,\frac{\Lambda^2\sqrt{\Lambda^2+M^2}}{\Lambda^2+M^2+\frac{1}{4}p^2}.\right\}\tag{10}$$

Making use of the covariant expansion

$$\left. \begin{aligned} \left(\Lambda^2 + M^2 + \frac{1}{4}p^2\right)^{-1} &= \left(\Lambda^2 + M^2 - \frac{1}{4}m^2\right)^{-1} \\ \times \left[1 + \frac{1}{4}\frac{(-p^2 - m^2)}{\left(\Lambda^2 + M^2 - \frac{1}{4}m^2\right)} \right. &+ \left. \frac{1}{16}\frac{(-p^2 - m^2)^2}{\left(\Lambda^2 + M^2 - \frac{1}{4}m^2\right)^2} + \cdots \right] \end{aligned} \right\} \quad (11)$$

we finally obtain the operator Φ introduced by (8) as a power series in the operator $(\square - m^2)/M^2$,

$$\Phi = \delta m^2 + \varepsilon(\square - m^2) + c^{(1)}\frac{\square - m^2}{M^2}(\square - m^2) + \ldots \quad (12)$$

It is convenient to express the induced source density $\langle I \rangle_0$ in terms of the external source density $I^{(e)}$ creating the external meson field due to

$$(\square - m^2)\, u = I^{(e)}, \quad (13)$$

and by (8) and (12) the expression for $\langle I \rangle_0$ is

$$\langle I \rangle_0 = \delta m^2 u + \varepsilon I^{(e)} + c^{(1)}\frac{\square - m^2}{M^2} I^{(e)} + \cdots . \quad (14)$$

The various constants introduced are easily obtained from (10) and (11). We get

$$\left. \begin{aligned} \delta m^2 &= -4\, g^2 (2\pi)^{-2} \int_0^\infty \frac{|G(\Lambda^2)|^2 \cdot \Lambda^2 \sqrt{\Lambda^2 + M^2}}{\Lambda^2 + M^2 - \frac{1}{4}m^2} d\Lambda \\ \varepsilon &= -\quad g^2 (2\pi)^{-2} \int_0^\infty \frac{|G(\Lambda^2)|^2 \cdot \Lambda^2 \sqrt{\Lambda^2 + M^2}}{\left(\Lambda^2 + M^2 - \frac{1}{4}m^2\right)^2} d\Lambda \\ c^{(1)} &= -\frac{1}{4} g^2 (2\pi)^{-2} \int_0^\infty \frac{|G(\Lambda^2)|^2 \cdot M^2 \Lambda^2 \sqrt{\Lambda^2 + M^2}}{\left(\Lambda^2 + M^2 - \frac{1}{4}m^2\right)^3} d\Lambda . \end{aligned} \right\} \quad (15)$$

Clearly, δm^2 represents the contribution to the square of the meson mass due to the interaction of the meson field with the nucleons. In fact, δm^2 in (15) is identical to (4.19). The induced density $\varepsilon I^{(e)}$ is also unobservable in principle and gives rise to a change of the coupling constant by an amount εg. Thus, the first observable term in the series is the third one. It will be seen from (15) that the numerical values of the expansion coefficients are highly sensible to the choice of the form factor. In the local limit, the two first of these diverge, δm^2 quadratically and ε logarithmically, while with the choice (4.18) of the form factor we obtain the finite and small corrections

$$\left.\begin{aligned} \frac{\delta m}{m} &= -\frac{2}{3\pi}\left(\frac{g^2}{4\pi}\right)\frac{m}{M}\alpha^{-3} \\ \varepsilon &= -\frac{1}{3\pi}\left(\frac{g^2}{4\pi}\right)\left(\frac{m}{M}\right)^3\alpha^{-3}, \end{aligned}\right\} \tag{16}$$

where we have neglected higher powers in $1/\lambda M$.

Here, α is the product of λ with the meson mass m. Also the value obtained for the constant $c^{(1)}$ is considerably reduced by the introduction of the form factor. In fact, in the local limit, we obtain

$$c^{(1)}_{\text{local}} = -\frac{1}{12\pi}\left(\frac{g^2}{4\pi}\right) \tag{17}$$

while, using the form factor (4.18), $c^{(1)}$ becomes

$$c^{(1)} = -\frac{1}{12\pi}\left(\frac{g^2}{4\pi}\right)\left(\frac{m}{M}\right)^3\alpha^{-3}. \tag{18}$$

The ratio of the two values

$$\frac{c^{(1)}}{c^{(1)}_{\text{local}}} = \left(\frac{m}{M}\right)^3\alpha^{-3} \tag{19}$$

may be expected to be small. Thus, in the present theory, $c^{(1)}_{\text{local}}$ does not represent the true vacuum polarization, contrary to what would be expected from a renormalization point of view. This is in accordance with the point of view that the difficulties in quantum field theory should be overcome by a modification of the theories in the high energy region.

It is seen from (15) that the main contribution to $c^{(1)}_{\text{local}}$ comes from the region $\Lambda \sim 2M$. Hence, in meson theory, the vacuum polarization should be considered as a high energy phenomenon, contrary to what is the case in electrodynamics where the main contribution to the induced current comes from distances of the order of the Compton wavelength of the electron. This distance must be expected to be large compared with the constant λ' which must be expected to occur in a convergent electron theory.

Added in proof. Professor W. PAULI has kindly pointed out to us that it is possible to construct a tensor $t_{\mu\nu}$ and a four vector j_μ having the properties that a) for $\lambda \to 0$ $t_{\mu\nu}$ and j_μ become identical with the usual expressions for the energy-momentum tensor and the four current of the field, respectively, and b) that $t_{\mu\nu}$ and j_μ satisfy the strict continuity equations $\partial_\nu t_{\mu\nu} = 0$ and $\partial_\mu j_\mu = 0$. As shown by Professor PAULI, this opens the interesting possibility to introduce a Hamiltonian formalism and, hence, to perform a canonical quantization of the theory. We are greatly indebted to Professor PAULI for many illuminating discussions and comments on the subject of this papir.

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Appendix A.

In this appendix*), we shall, for the purpose of reference, give the definition of the various Green's functions introduced in the text and their Fourier expansions. The singular function Δ can be defined as Green's function solving the initial value problem of the homogeneous wave equation. Let us consider that solution $\Phi(x, \sigma)$ of the equation

$$(\Box - \varkappa^2)\,\Phi(x, \sigma) = 0 \tag{1}$$

which, together with its normal derivative, is given on a space-like surface σ. Writing the solution in the form of a surface integral

$$\Phi(x, \sigma) = \int_\sigma \{\Delta(x - x')\,\partial'_\mu \Phi(x', \sigma) - \Phi(x', \sigma)\,\partial'_\mu \Delta(x - x')\}\, d\sigma_\mu, \tag{2}$$

$\Delta(x)$ obviously must satisfy

$$\left.\begin{aligned} (\Box - \varkappa^2)\,\Delta(x) &= 0 \\ \Delta(x) &= 0,\ x_\mu x_\mu > 0 \\ \int_\sigma \partial_\mu \Delta(x)\, d\sigma_\mu &= 1 \end{aligned}\right\} \tag{3}$$

for any σ including the origin.

To solve the same boundary value problem of the inhomogeneous equation

$$(\Box - \varkappa^2)\,\Phi(x) = I(x) \tag{4}$$

we introduce one more Green's function $\Delta^\sigma(x, x')$ satisfying

* This appendix and the following contain no new results. For details and proofs the reader is referred to (1) and (10).

$$\left.\begin{aligned}(\square - \varkappa^2)\,\Delta^{\sigma}(x, x') &= -\delta(x - x')\\ \Delta^{\sigma}(x/\sigma, x') &= 0\\ \eta_{\mu}\partial_{\mu}\,\Delta^{\sigma}(x/\sigma, x') &= 0,\end{aligned}\right\}\tag{5}$$

where we have used the notation x/σ to indicate a point x lying on the surface σ. n_{μ} is the unit normal to σ in the point x/σ. The solution of the mentioned boundary value problem is then

$$\Phi(x) = \Phi(x, \sigma) - \int \Delta^{\sigma}(x, x')\, I(x')\, dx', \tag{6}$$

where the free field $\Phi(x, \sigma)$, coinciding with $\Phi(x)$ on σ, is given by (2). Taking in (5) for fixed x', σ in the infinite past, we obtain the retarded Green's function

$$\Delta^{\mathrm{ret}}(x - x') = \lim_{\sigma \to -\infty} \Delta^{\sigma}(x, x') \tag{7}$$

satisfying

$$\left.\begin{aligned}(\square - \varkappa^2)\,\Delta^{\mathrm{ret}}(x - x') &= -\delta(x - x')\\ \lim_{x_0 \to -\infty} \Delta^{\mathrm{ret}}(x - x') &= 0\\ \lim_{x_0 \to -\infty} \partial_0 \Delta^{\mathrm{ret}}(x - x') &= 0.\end{aligned}\right\}\tag{8}$$

Formally, Δ^{ret} solves the initial value problem, where the asymptotic form of $\Phi(x)$ and its derivative in the time direction are given at the infinite past. In the same way, we can define

$$\Delta^{\mathrm{adv}}(x - x') = \lim_{o \to +\infty} \Delta^{\sigma}(x, x') \tag{9}$$

satisfying

$$\left.\begin{aligned}(\square - \varkappa^2)\,\Delta^{\mathrm{adv}}(x - x') &= -\delta(x - x')\\ \lim_{x_0 \to +\infty} \Delta^{\mathrm{adv}}(x - x') &= 0\\ \lim_{x_0 \to +\infty\, \Delta} \partial_0 \Delta^{\mathrm{adv}}(x - x') &= 0.\end{aligned}\right\}\tag{10}$$

Starting from the thus defined Green's functions Δ, Δ^{ret} and Δ^{adv}, we can define various other singular functions satisfying either the homogeneous equation

$$(\Box - \varkappa^2)\,\Phi(x) = 0 \tag{11}$$

or the inhomogeneous equation

$$(\Box - \varkappa^2)\,\Phi(x) = -\delta(x). \tag{12}$$

It is clear that the positive and negative frequency parts of $\varDelta$

$$\varDelta^{(+)} = \text{positive frequency part of } \varDelta \tag{13}$$

$$\varDelta^{(-)} = \text{negative frequency part of } \varDelta \tag{14}$$

satisfy (11). The same holds for the function

$$\varDelta^{(1)} = i\,(\varDelta^{(+)} - \varDelta^{(-)}). \tag{15}$$

On the other hand, $\overline{\varDelta}$ defined by

$$\overline{\varDelta} = \frac{1}{2}(\varDelta^{\mathrm{adv}} + \varDelta^{\mathrm{ret}}) \tag{16}$$

is a solution of (12).

If we introduce the characteristic functions

$$\left.\begin{aligned} \varepsilon(x) &= \operatorname{sign} x_0 \\ \varepsilon(\sigma, x) &= \begin{cases} -1 & \text{for } x \text{ on the future side of } \sigma \\ +1 & \text{for } x \text{ on the past side of } \sigma \end{cases} \end{aligned}\right\} \tag{17}$$

the following relations can be shown to hold among the various functions introduced.

$$\varDelta = \varDelta^{(+)} + \varDelta^{(-)} \tag{18}$$

$$\overline{\varDelta} = -\frac{1}{2}\varepsilon(x)\,\varDelta(x) \tag{19}$$

$$\varDelta = \varDelta^{\mathrm{adv}} - \varDelta^{\mathrm{ret}} \tag{20}$$

$$\varDelta^{\mathrm{adv}} = \overline{\varDelta} + \frac{1}{2}\varDelta = -\frac{\varepsilon(x) - 1}{2}\varDelta \tag{21}$$

$$\varDelta^{\mathrm{ret}} = \overline{\varDelta} - \frac{1}{2}\varDelta = -\frac{\varepsilon(x) + 1}{2}\varDelta \tag{22}$$

$$\varDelta^{\sigma}(x, x') = -\frac{1}{2}\{\varepsilon(x - x') - \varepsilon(\sigma, x')\}\,\varDelta(x - x'). \tag{23}$$

From the well known Fourier expansions of Δ and $\overline{\Delta}$ and the relations given above, one can easily deduce the expansions for the other functions. We have

$$\Delta = -i(2\pi)^{-3}\int \varepsilon(k)\,\delta(k^2+\varkappa^2)\,e^{ikx}dk \tag{24}$$

$$\Delta^{(+)} = -i(2\pi)^{-3}\int \frac{1+\varepsilon(k)}{2}\,\delta(k^2+\varkappa^2)\,e^{ikx}dk \tag{25}$$

$$\Delta^{(-)} = i(2\pi)^{-3}\int \frac{1-\varepsilon(k)}{2}\,\delta(k^2+\varkappa^2)\,e^{ikx}dk \tag{26}$$

$$\Delta^{(1)} = (2\pi)^{-3}\int \delta(k^2+\varkappa^2)\,e^{ikx}dk \tag{27}$$

$$\overline{\Delta} = (2\pi)^{-4}\int \frac{1}{k^2+\varkappa^2}\,e^{ikx}dk \tag{28}$$

$$\Delta^{\mathrm{ret}} = (2\pi)^{-4}\int \left\{\frac{1}{k^2+\varkappa^2}+i\pi\varepsilon(k)\,\delta(k^2+\varkappa^2)\right\} e^{ikx}dk \tag{29}$$

$$\Delta^{\mathrm{adv}} = (2\pi)^{-4}\int \left\{\frac{1}{k^2+\varkappa^2}-i\pi\varepsilon(k)\,\delta(k^2+\varkappa^2)\right\} e^{ikx}dk. \tag{30}$$

Here, $\frac{1}{k^2+\varkappa^2}$ should be understood as Cauchy's principal value, so that, for instance,

$$\Delta^{\mathrm{ret}} = (2\pi)^{-4}\int_{-\infty}^{\infty} d^{(3)}\vec{k}\int_{C^{\mathrm{ret}}} dk_0\,\frac{1}{k^2+\varkappa^2}\,e^{ikx}, \tag{31}$$

where C^{ret} is taken along the k_0-axes below the poles at $k_0 = \pm\sqrt{\vec{k}^2+\varkappa^2}$. In this form it can easily be verified that Δ^{ret} has the required asymptotic form.

Let us denote any of the Green's functions introduced above by $\Delta^?$. The corresponding Green's functions belonging to the Dirac equation are then defined as

$$S^? = (\gamma_\mu\partial_\mu-\varkappa)\,\Delta^?. \tag{32}$$

For completeness, we give the expansions for the S-functions

$$S = (2\pi)^{-3}\int (\gamma_\mu k_\mu + i\varkappa)\,\varepsilon(k)\,\delta(k^2+\varkappa^2)\,e^{ikx}dk \tag{33}$$

$$^{(+)} = (2\,\pi)^{-3} \int (\gamma_\mu k_\mu + i\varkappa)\, \frac{1+\varepsilon(k)}{2}\, \delta(k^2+\varkappa^2)\, e^{ikx} dk \tag{34}$$

$$^{(-)} = -(2\,\pi)^{-3} \int (\gamma_\mu k_\mu + i\varkappa)\, \frac{1-\varepsilon(k)}{2}\, \delta(k^2+\varkappa^2)\, e^{ikx} dk \tag{35}$$

$$^{(1)} = i\,(2\,\pi)^{-3} \int (\gamma_\mu k_\mu + i\varkappa)\, \delta(k^2+\varkappa^2)\, e^{ikx} dk \tag{36}$$

$$= i\,(2\,\pi)^{-4} \int \frac{\gamma_\mu k_\mu + i\varkappa}{k^2+\varkappa^2}\, e^{ikx} dk \tag{37}$$

$$^{\mathrm{ret}} = i\,(2\,\pi)^{-4} \int (\gamma_\mu k_\mu + i\varkappa) \left\{ \frac{1}{k^2+\varkappa^2} + i\pi\varepsilon(k)\, \delta(k^2+\varkappa^2) \right\} e^{ikx} dk \tag{38}$$

$$^{\mathrm{adv}} = i\,(2\,\pi)^{-4} \int (\gamma_\mu k_\mu + i\varkappa) \left\{ \frac{1}{k^2+\varkappa^2} - i\pi\varepsilon(k)\, \delta(k^2+\varkappa^2) \right\} e^{ikx} dk\,. \tag{39}$$

Appendix B.

In this appendix, we shall derive the various matrix elements needed in the calculation of the S-matrix. All field variables considered here are in-field variables and will be written without the subscript "in". For simplicity, we also use the notation of § 3, i. e. instead of x', x'', ..., we write 1, 2, ... The spinor variables can, in a relativistically invariant way, be decomposed into a positive and a negative frequency part

$$\psi = \psi^{(+)} + \psi^{(-)}, \quad \overline{\psi} = \overline{\psi}^{(+)} + \overline{\psi}^{(-)}. \tag{1}$$

From the well known commutation relations for $\overline{\psi}$ and ψ one immediately finds that the only non-vanishing anticommutators are

$$\left.\begin{aligned} \left\{\psi_\alpha^{(+)}(3), \overline{\psi}_\beta^{(-)}(1)\right\} &= -iS_{\alpha\beta}^{(+)}(3-1) \\ \left\{\psi_\alpha^{(-)}(3), \overline{\psi}_\beta^{(+)}(1)\right\} &= -iS_{\alpha\beta}^{(-)}(3-1). \end{aligned}\right\} \tag{2}$$

The operation of any positive frequency operator on a state of the nucleon field lowers the energy of the system and the operation of the ψ-function lowers the value of the quantity ΔN*) while the negative frequency operators and $\overline{\psi}$ increase the energy and ΔN, respectively. Accordingly, $\overline{\psi}^{(-)}$ creates nucleons, $\psi^{(-)}$ creates antinucleons, while $\overline{\psi}^{(+)}$ and $\psi^{(+)}$ annihilate antinucleons and nucleons, respectively. The vacuum state vector $|0\rangle$, defined as the state in which no particles are present, then satisfies

$$\psi^{(+)}|0\rangle = 0, \quad \overline{\psi}^{(+)}|0\rangle = 0, \tag{3}$$

and the Hermitian conjugate equations

$$\langle 0|\overline{\psi}^{(-)} = 0, \quad \langle 0|\psi^{(-)} = 0. \tag{4}$$

*) $\Delta N = \int \psi^\dagger \psi d^{(3)}\vec{x}$.

(2), (3), and (4) allow us to calculate the vacuum expectation value of any product of $\overline{\psi}$ and ψ functions occurring in the S-matrix. For instance,

$$\langle 0 \,|\, \overline{\psi}_\alpha (1)\, \psi_\beta (3) \,|\, 0 \rangle = - i S^{(-)}_{\beta\alpha} (3-1). \tag{5}$$

Using (1), the vacuum definitions (3) and (4), and the relation (2), the proof is straightforward

$$\begin{aligned} \langle 0 \,|\, \overline{\psi}_\alpha (1)\, \psi_\beta (3) \,|\, 0 \rangle &= \langle 0 \,|\, \overline{\psi}^{(+)}_\alpha (1)\, \psi^{(-)}_\beta (3) \,|\, 0 \rangle \\ &= \langle 0 \,|\, \{ \overline{\psi}^{(+)}_\alpha (1), \quad \psi^{(-)}_\beta (3) \} \,|\, 0 \rangle \\ &= - i S^{(-)}_{\beta\alpha} (3-1), \end{aligned}$$

if we assume that the vacuum state vector is normalized. In a similar way, we can show that

$$\left. \begin{aligned} &\langle 0 \,|\, \overline{\psi}_\alpha (1)\, \overline{\psi}_\beta (4)\, \psi_\gamma (6)\, \psi_\delta (3) \,|\, 0 \rangle \\ = \; & S^{(-)}_{\gamma\alpha} (6-1)\, S^{(-)}_{\delta\beta} (3-4) - S^{(-)}_{\gamma\beta} (6-4)\, S^{(-)}_{\delta\alpha} (3-1). \end{aligned} \right\} \tag{6}$$

To define states in which nucleons are present, it will be convenient to work in the momentum representation. We introduce the following notations: $v^{(+)} (\sigma, \vec{P}) \exp (i P^{(+)} x)$, and $v^{(-)} (\sigma, \vec{P}) \exp (i P^{(-)} x)$ are the one particle eigenstates of energy and momentum satisfying the Dirac equation and corresponding to positive and negative states of energy, respectively. If the amplitudes $v^{(+)}$ and $v^{(-)}$ are normalized, the expansion coefficients a defined by

$$\left. \begin{aligned} \psi^{(+)} (x) &= (2\pi)^{-\frac{3}{2}} \sum_\sigma \int d^{(3)} \vec{P} \cdot v^{(+)} (\sigma, \vec{P})\, e^{i P^{(+)} x} \; a^{(+)} (\sigma, \vec{P}) \\ \overline{\psi}^{(-)} (x) &= (2\pi)^{-\frac{3}{2}} \sum_\sigma \int d^{(3)} \vec{P} \cdot \overline{v}^{(-)} (\sigma, \vec{P})\, e^{-i P^{(+)} x} \; \overline{a}^{(-)} (\sigma, \vec{P}) \\ \psi^{(-)} (x) &= (2\pi)^{-\frac{3}{2}} \sum_\sigma \int d^{(3)} \vec{P} \cdot v^{(-)} (\sigma, \vec{P})\, e^{i P^{(-)} x} \; a^{(-)} (\sigma, \vec{P}) \\ \overline{\psi}^{(+)} (x) &= (2\pi)^{-\frac{3}{2}} \sum_\sigma \int d^{(3)} \vec{P} \cdot \overline{v}^{(+)} (\sigma, \vec{P})\, e^{-i P^{(-)} x} \; \overline{a}^{(+)} (\sigma, \vec{P}) \end{aligned} \right\} \tag{7}$$

satisfy the following commutation relations (only the nonvanishing anticommutators are written)

$$\left.\begin{aligned} \left\{\overline{a}^{(+)}(\sigma'', \vec{P}''), a^{(-)}(\sigma', \vec{P}')\right\} &= \delta_{\sigma''\sigma'}\delta(\vec{P}'' - \vec{P}') \\ \left\{\overline{a}^{(-)}(\sigma'', \vec{P}''), a^{(+)}(\sigma', \vec{P}')\right\} &= \delta_{\sigma''\sigma'}\delta(\vec{P}'' - \vec{P}'). \end{aligned}\right\} \tag{8}$$

In (7), $P^{(+)}$ is short for $(\vec{P}, i\sqrt{\vec{P}^2 + M^2})$ and $P^{(-)}$ for $(\vec{P}, -i\sqrt{\vec{P}^2 + M^2})$ while σ is the spin quantum number. It is easily seen from (7) that

$$\overline{a}^{(+)} = a^{(-)\dagger}, \quad \overline{a}^{(-)} = a^{(+)\dagger}. \tag{9}$$

The one particle states are now defined in the following way

$$\left.\begin{aligned} |\sigma P^{(+)}\rangle &= \overline{a}^{(-)}(\sigma, \vec{P})|0\rangle, \quad \langle\sigma P^{(+)}| = \langle 0|a^{(+)}(\sigma, \vec{P}) \\ |\sigma P^{(-)}\rangle &= a^{(-)}(\sigma, \vec{P})|0\rangle, \quad \langle\sigma P^{(-)}| = \langle 0|\overline{a}^{(+)}(\sigma, \vec{P}). \end{aligned}\right\} \tag{10}$$

By this definition the states with one particle present are automatically normalized. For instance,

$$\begin{aligned} \langle\sigma P^{(+)}|\sigma' P^{(+)\prime}\rangle &= \langle 0|a^{(+)}(\sigma, \vec{P})\,\overline{a}^{(-)}(\sigma', \vec{P}')|0\rangle \\ &= \langle 0|\left\{a^{(+)}(\sigma, \vec{P}), \overline{a}^{(-)}(\sigma', \vec{P}')\right\}|0\rangle \\ &= \delta_{\sigma\sigma'}\,\delta(\vec{P} - \vec{P}')\langle 0|0\rangle \\ &= \delta_{\sigma\sigma'}\,\delta(\vec{P} - \vec{P}'). \end{aligned}$$

If an annihilation operator $a^{(+)}$ is applied to $|\sigma P^{(+)}\rangle$, we obtain

$$a^{(+)}(\sigma', \vec{P}')|\sigma P^{(+)}\rangle = \delta_{\sigma\sigma'}\delta(\vec{P} - \vec{P}')|0\rangle. \tag{11}$$

In the same way, states with two particles present are defined as

$$|\sigma'' P^{(+)\prime\prime}, \sigma' P^{(+)\prime}\rangle = \overline{a}^{(-)}(\sigma'', \vec{P}'')\,\overline{a}^{(-)}(\sigma', \vec{P}')|0\rangle. \tag{12}$$

By this definition the states are automatically normalized and antisymmetric in the two particles, i. e.

$$|\sigma'' P^{(+)\prime\prime}, \sigma' P^{(+)\prime}\rangle = -|\sigma' P^{(+)\prime}, \sigma'' P^{(+)\prime\prime}\rangle \tag{13}$$

in accordance with the exclusion principle.

If an annihilation operator is applied to the state (12) one obtains

$$\left.\begin{aligned} a^{(+)}(\vec{P}''', \sigma''') \,|\, \sigma'' P^{(+)''}, \sigma' P^{(+)'} \rangle = \\ = \delta_{\sigma''' \sigma''} \delta(\vec{P}''' - \vec{P}'') \,|\, \sigma' P^{(+)'} \rangle - \delta_{\sigma''' \sigma'} \delta(\vec{P}''' - \vec{P}') \,|\, \sigma'' P^{(+)''} \rangle, \end{aligned}\right\} \quad (14)$$

a relation which can be verified most easily in the standard way by pushing the annihilation operator through to the right, using (8), (3), and (10).

We can now derive the matrix elements of the one particle part of the operator $\overline{\psi}_\alpha(1)\,\psi_\beta(6)$ obtained by subtraction of the vacuum expectation value times the unit operator, i. e.

$$\left[\overline{\psi}_\alpha(1)\,\psi_\beta(6)\right]_{(1)} = \overline{\psi}_\alpha(1)\,\psi_\beta(6) - \langle 0 \,|\, \overline{\psi}_\alpha(1)\,\psi_\beta(6) \,|\, 0 \rangle. \quad (15)$$

For instance, if the initial and final states both are nucleon states, we obtain

$$\left.\begin{aligned} \langle \sigma'' P^{(+)''} \,|\, \left[\overline{\psi}_\alpha(1)\,\psi_\beta(6)\right]_{(1)} \,|\, \sigma' P^{(+)'} \rangle = \\ = (2\pi)^{-3}\, \overline{v}_\alpha^{(-)}(\sigma'', \vec{P}'')\, v_\beta^{(+)}(\sigma', \vec{P}') \exp i\,(P^{(+)'} 6 - P^{(+)''} 1). \end{aligned}\right\} \quad (16)$$

To prove this, we first remark that

$$\begin{aligned} \langle \sigma'' P^{(+)''} \,|\, \overline{\psi}_\alpha(1)\,\psi_\beta(6) \,|\, \sigma' P^{(+)'} \rangle = \\ = \langle \sigma'' P^{(+)''} \,|\, \overline{\psi}_\alpha^{(+)}(1)\,\psi_\beta^{(-)}(6) \,|\, \sigma' P^{(+)'} \rangle \\ + \langle \sigma'' P^{(+)''} \,|\, \overline{\psi}_\alpha^{(-)}(1)\,\psi_\beta^{(+)}(6) \,|\, \sigma' P^{(+)'} \rangle, \end{aligned}$$

which is a consequence of the fact that terms containing two creation or two annihilation operators vanish when the number of particles is the same in the two states considered. The first of the terms on the right hand side is easily identified as the matrix element of the operator subtracted in (15), and the second becomes identical with the right hand side of (16) if one introduces the expansion (7) of $\overline{\psi}^{(-)}$ and $\psi^{(+)}$ and uses (11).

Similarly, we find

$$\left.\begin{aligned} \langle \sigma'' P^{(+)''} \,|\, \left[\overline{\psi}_\alpha(1)\,\overline{\psi}_\beta(4)\,\psi_\gamma(6)\,\psi_\delta(3)\right]_{(1)} \,|\, \sigma' P^{(+)'} \rangle = \\ -i(2\pi)^{-3} \Big\{ \overline{v}_\beta^{(-)}(\sigma'', \vec{P}'')\, v_\gamma^{(+)}(\sigma', \vec{P}') \exp i\,(P^{(+)'} 6 - P^{(+)''} 4) \cdot S_{\delta\alpha}^{(-)}(3-1) \\ + \overline{v}_\alpha^{(-)}(\sigma'', \vec{P}'')\, v_\delta^{(+)}(\sigma', \vec{P}') \exp i\,(P^{(+)'} 3 - P^{(+)''} 1) \cdot S_{\gamma\beta}^{(-)}(6-4) \\ - \overline{v}_\beta^{(-)}(\sigma'', \vec{P}'')\, v_\delta^{(+)}(\sigma', \vec{P}') \exp i\,(P^{(+)'} 3 - P^{(+)''} 4) \cdot S_{\gamma\alpha}^{(-)}(6-1) \\ - \overline{v}_\alpha^{(-)}(\sigma'', \vec{P}'')\, v_\gamma^{(+)}(\sigma', \vec{P}') \exp i\,(P^{(+)'} 6 - P^{(+)''} 1) \cdot S_{\delta\beta}^{(-)}(3-4) \Big\} \end{aligned}\right\} \quad (17)$$

Finally, using the same type of arguments, we obtain the matrix elements of the two particle part of the product of four ψ-functions. In the derivation, due regard must be paid to the minus signs introduced by the annihilation processes as illustrated by (14). The result is

$$\left.\begin{aligned}
&\langle P''\sigma'', \sigma^{\mathrm{iv}} P^{(+)\mathrm{iv}} \left| \left[\overline{\psi}_\alpha(1)\,\overline{\psi}_\beta(4)\,\psi_\gamma(6)\,\psi_\delta(3)\right]_{(2)} \right| \sigma' P^{(+)\prime}, \sigma''' P^{(+)\prime\prime\prime} \rangle = \\
&= (2\pi)^{-6}\Big\{\overline{v}^{(-)}_\alpha(\sigma^{\mathrm{iv}}, \vec{P}^{\mathrm{iv}})\,\overline{v}^{(-)}_\beta(\sigma^{\mathrm{iv}}, \vec{P}^{\mathrm{iv}})\,v^{(+)}_\gamma(\sigma', \vec{P}')\,v^{(+)}_\delta(\sigma''', \vec{P}''') \\
&\qquad \times \exp i\{P^{(+)\prime}6 + P^{(+)\prime\prime\prime}3 - P^{(+)\prime\prime}1 - P^{(+)\mathrm{iv}}4\} \\
&\qquad - \overline{v}^{(-)}_\beta(\sigma'', \vec{P}'')\,\overline{v}^{(-)}_\alpha(\sigma^{\mathrm{iv}}, \vec{P}^{\mathrm{iv}})\,v^{(+)}_\gamma(\sigma', \vec{P}')\,v^{(+)}_\delta(\sigma''', \vec{P}''') \\
&\qquad \times \exp i\{P^{(+)\prime}6 + P^{(+)\prime\prime\prime}3 - P^{(+)\prime\prime}4 - P^{(+)\mathrm{iv}}1\} \\
&\qquad - \overline{v}^{(-)}_\alpha(\sigma'', \vec{P}'')\,\overline{v}^{(-)}_\beta(\sigma^{\mathrm{iv}}, \vec{P}^{\mathrm{iv}})\,v^{(+)}_\delta(\sigma', \vec{P}')\,v^{(+)}_\gamma(\sigma''', \vec{P}'''') \\
&\qquad \times \exp i\{P^{(+)\prime}3 + P^{(+)\prime\prime\prime}6 - P^{(+)\prime\prime}1 - P^{(+)\mathrm{iv}}4\} \\
&\qquad + \overline{v}^{(-)}_\beta(\sigma'', \vec{P}'')\,\overline{v}^{(-)}_\alpha(\sigma^{\mathrm{iv}}, \vec{P}^{\mathrm{iv}})\,v^{(+)}_\delta(\sigma', \vec{P}')\,v^{(+)}_\gamma(\sigma''', \vec{P}''') \\
&\qquad \times \exp i\{P^{(+)\prime}3 + P^{(+)\prime\prime\prime}6 - P^{(+)\prime\prime}4 - P^{(+)\mathrm{iv}}1\}\Big\}.
\end{aligned}\right\} \quad (18$$

The corresponding results for the free meson field are the following. The meson wave functions can be decomposed into a positive and a negative frequency part $u^{(+)}$ and $u^{(-)}$, where $u^{(-)}$ is the creation operator and $u^{(+)}$ the annihilation operator of the field. The meson vacuum is defined by

$$u^{(+)}\left|0\right\rangle = 0, \quad \left\langle o\right| u^{(-)} = 0 \tag{19}$$

and the non-vanishing commutators are

$$\left.\begin{aligned}
[u^{(+)}(2), u^{(-)}(5)] &= i\Delta^{(+)}(2-5) \\
[u^{(-)}(2), u^{(+)}(5)] &= i\Delta^{(-)}(2-5)
\end{aligned}\right\} \tag{20}$$

in accordance with $\Delta^{(+)}(x) = -\Delta^{(-)}(-x)$. From (19) and (20) we get immediately the following vacuum expectation value used in the text

$$\left\langle 0 \right| u(2)\,u(5) \left| 0 \right\rangle = i\Delta^{(+)}(2-5). \tag{21}$$

Introducing the quantity

$$\omega(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}, \quad p = (\vec{p}, i\omega) \tag{22}$$

$u^{(-)}$ and $u^{(+)}$ can be expanded

$$\left.\begin{aligned} u^{(+)} &= (2\pi)^{-\frac{3}{2}} \int \frac{1}{\sqrt{2\omega}} \cdot b(\vec{p})\, e^{ipx} d^{(3)}\vec{p} \\ u^{(-)} &= (2\pi)^{-\frac{3}{2}} \int \frac{1}{\sqrt{2\omega}} \cdot b^{\dagger}(\vec{p})\, e^{-ipx} d^{(3)}\vec{p} \end{aligned}\right\} \tag{23}$$

and the b's are seen to satisfy the commutation relations

$$[b(\vec{p}\,'), b^{\dagger}(\vec{p})] = \delta(\vec{p}\,' - \vec{p}). \tag{24}$$

A state with one meson present of momentum $\vec{p}$ is defined as

$$|\vec{p}\rangle = b^{\dagger}(\vec{p})\,|0\rangle \tag{25}$$

and it follows that the one particle part of $u(2)\,u(5)$ has the matrix elements

$$\left.\begin{aligned} &\langle \vec{p}\,'' \,|\, [u(2)\,u(5)]_{(1)} \,|\, \vec{p}\,' \rangle = \\ &= \frac{1}{2}(2\pi)^{-3} \left\{ \omega(\vec{p}\,'')\,\omega(\vec{p}\,') \right\}^{-\frac{1}{2}} \\ &\times \left[e^{i\{p'5 - p''2\}} + e^{i\{p'2 - p''5\}} \right]. \end{aligned}\right\} \tag{26}$$

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